Published by Institute of Physics Publishing for SISSA/ISAS
Received: *October 21, 2004*
Accepted: *December 22, 2004*

Elliptic Calabi-Yau threefolds with $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines

Volker Braun, Burt A. Ovrut and Tony Pantev

David Rittenhouse Laboratory, University of Pennsylvania
209 S. 33rd Street, Philadelphia, PA 19104, U.S.A.
E-mail: vbraun@physics.upenn.edu, ovrut@elcapitan.hep.upenn.edu, tpantev@math.upenn.edu

René Reinbacher

Department of Physics and Astronomy, Rutgers University
136 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A.
E-mail: rreinb@physics.rutgers.edu

Abstract: A torus fibered Calabi-Yau threefold with first homotopy group $\mathbb{Z}_3 \times \mathbb{Z}_3$ is constructed as a free quotient of a fiber product of two dP_9 surfaces. Calabi-Yau threefolds of this type admit $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines. In conjunction with SU(4) holomorphic vector bundles, such vacua lead to anomaly free, three generation models of particle physics with a right handed neutrino and a $\mathrm{U}(1)_{B-L}$ gauge factor, in addition to the $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ standard model gauge group. This factor helps to naturally suppress nucleon decay. The moduli space and Dolbeault cohomology of the threefold is also discussed.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Differential and Algebraic Geometry.

Contents

1. Introduction

A long standing question has been whether or not there exist, within the context of both weakly [1, 2] and strongly coupled [3, 4] $E_8 \times E_8$ heterotic string theory, vacua that accurately describe low energy particle physics. One conceptually simple approach has been to compactify heterotic string theory on a smooth Calabi-Yau threefold X which admits a gauge connection satisfying the hermitian Yang-Mills equations.

It was shown in [5, 6] that this latter requirement is equivalent to demonstrating the existence of a stable, holomorphic vector bundle V on X. In early work, V was chosen to be the tangent bundle of X; that is, $V = TX$. This so-called standard embedding is reviewed in [7, 8]. While interesting, this choice is extremely restrictive, fixing V to be one out of an enormous number of possible vector bundles. Note, for example, that since X is a Calabi-Yau threefold, the structure group of TX must be SU(3). Thus, the low energy gauge group associated with $V = TX$, ignoring possible Wilson lines, is E_6, the commutant of SU(3) in E_8. Although E_6 is a possible grand unified group, other choices, such as SU(5) or Spin(10), would be physically more interesting.

It is important, therefore, to construct stable holomorphic vector bundles $V \neq TX$. Recently, it was shown how to obtain such bundles over simply connected Calabi-Yau threefolds that are elliptically fibered [9, 10]. This work was extended, within the context of heterotic M-Theory [11]–[15], in [10, 16]. These new vector bundles have arbitrary structure groups, such as SU(5) and SU(4). These lead to a wide range of unified gauge groups, including SU(5) and Spin(10). Many of the physical properties of these generalized vacua have been studied, such as the moduli space of the associated $M5$-branes [17], small instanton phase transitions [18], fluxes [19]–[24], supersymmetry breaking [25], the moduli space of the vector bundle V [26, 27], and non-perturbative superpotentials [26, 28, 29, 30]. Recently, it was shown how to compute the sheaf cohomology of V, as well as that of its tensor products. This determines the complete particle spectrum [31, 32]. These vacua also underly the theory of brane universes [11]–[15] and ekpyrotic cosmology [33]–[36] in strongly coupled heterotic strings.

However, heterotic vacua that have the standard model gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ as a factor need additional ingredients. First of all, the Calabi-Yau threefold must have a non-trivial first homotopy group $\pi_1(X) \neq 1$. Only such manifolds will admit Wilson lines [37]–[41]. One must then construct stable, holomorphic vector bundles on these spaces which, in conjunction with Wilson lines, will reduce the gauge group so as to include $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ as a factor. Torus fibered Calabi-Yau threefolds with $\pi_1(X) = \mathbb{Z}_2$ were considered in [42]–[45]. These manifolds admit stable, holomorphic vector bundles with structure group SU(5). Together with the $\mathbb{Z}_2$ Wilson lines, such vacua produce anomaly free theories with three generations, and gauge group exactly $\mathrm{SU}(3)_C \times$

$SU(2)_L \times U(1)_Y$. In recent work, the complete cohomology ring and, hence, the spectrum of such vacua was computed. These models were found to have at least one pair of Higgs doublets, but contain exotic supermultiplets as well. This work was generalized in [46]–[48] to torus fibered Calabi-Yau threefolds with $\pi_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_2$. These were shown to admit stable, holomorphic vector bundles with structure group SU(4).

The recent discovery that neutrinos have a nonzero mass [49] has made low energy theories based on spontaneously broken Spin(10) very attractive. There are several reasons for this. First, such theories naturally contain a right handed neutrino. Second, each family of quarks and leptons is unified within a single $\underline{\mathbf{16}}$, the spin representation of Spin(10). A third compelling reason is that these theories can contain an extra $U(1)_{B-L}$ gauge factor, which greatly helps to suppress the nucleon decay rate. For all these reasons, it would seem desirable to construct heterotic string vacua of this type. In this paper, we are taking the first steps in this direction. We begin by demonstrating that the simplest Wilson line that breaks Spin(10) to the standard model with an extra $U(1)_{B-L}$ requires a $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group. We then systematically construct torus fibered Calabi-Yau threefolds with this fundamental group. Several properties of these geometries will also be computed, including the Dolbeault cohomology groups. Stable, holomorphic vector bundles on these Calabi-Yau threefolds will be presented elsewhere [50]

The paper is organized as follows. First, in section 2, we review how a $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson line can break a Spin(10) gauge group down to the standard model plus a $U(1)_{B-L}$ factor. The general theory and the intermediate Spin(10) gauge group is introduced in subsection 2.1. Then, in 2.2 and 2.3 we successively consider the effect of two distinct $\mathbb{Z}_3$ Wilson lines. These two effects are combined in subsection 2.4, and the correct low energy gauge group is obtained.

In section 3, we analyze how one can implement this symmetry breaking within heterotic string theory. This imposes various restrictions on the compactification manifold. We will choose the Calabi-Yau threefold to be the fiber product of two dP_9 surfaces.

It follows that one must study dP_9 surfaces whose automorphism group contains a $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup. This will be done in section 4. First, the general topology of dP_9 surfaces is reviewed in subsection 4.1. We then study the general form of $\mathbb{Z}_3 \times \mathbb{Z}_3$ actions on these surfaces using the Mordell-Weil group and its bilinear form. These are introduced in subsections 4.2 and 4.3 and then applied to the dP_9 surfaces in 4.4. There are only a small number of allowed Mordell-Weil groups, and we use these to list all dP_9 surfaces having the desired properties. Their moduli are also discussed. Finally, in subsection 4.5 we investigate which surfaces yield fiber products with free $\mathbb{Z}_3 \times \mathbb{Z}_3$ actions. We conclude that only a single 1-parameter family is appropriate.

This 1-parameter family of dP_9 surfaces is constructed in section 5. To begin with, the Weierstrass model of an elliptic surface is reviewed in subsection 5.1. Then, we write down the Weierstrass equation for the desired dP_9 surface in subsection 5.2, thus establishing its existence. We check explicitly that it has the correct singular fibers and the desired group action. For completeness, the surface is also described as a pencil of cubics in subsection 5.3.

It remains to study the homology of this dP_9 surface and the induced action on the homology of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ automorphism. This is the aim of section 6. First, in subsection 6.1, we investigate how the singular fibers in the dP_9 surface intersect the sections of the elliptic fibration. Using this information, the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on the homology of the dP_9 surface is determined in subsection 6.2. In 6.3, the invariant part of the homology is finally computed.

Having studied these dP_9 surfaces in detail, we construct in section 7 the desired Calabi-Yau threefold. The fiber product of two such dP_9 surfaces is a Calabi-Yau manifold $\widetilde{X}$. Furthermore, if one makes the correct identifications in the fiber product, as discussed in subsection 7.1, then there exists a free $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ action on $\widetilde{X}$. The homology of $\widetilde{X}$ and the G action on it are discussed in subsection 7.2. In 7.3, we form the quotient $X = \widetilde{X}/G$. This is a Calabi-Yau threefold with fundamental group $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$, as desired. Its Hodge diamond is then computed and its moduli are discussed.

2. Breaking to the standard model gauge group

2.1 Symmetry breaking generalities

In this paper, we consider Calabi-Yau threefolds X with non-vanishing first homotopy group[1] $\pi_1(X)$. It is further assumed that these spaces admit stable, holomorphic vector bundles V with structure group

$$G_V = \mathrm{SU}(4) \subset E_8\,. \tag{2.1}$$

Specifying X and V determines a vacuum of $E_8 \times E_8$ heterotic string theory. This vacuum has, at low energies, $\mathcal{N} = 1$ supersymmetry in $\mathbb{R}^4$ with gauge symmetry

$$H_{\mathrm{SU}(4)} = \mathrm{Spin}(10) \tag{2.2}$$

in the observable sector. The low energy gauge group $H_{\mathrm{SU}(4)}$ is the commutant of SU(4) in E_8.

Since $\pi_1(X)$ is non-trivial, one can consider, in addition to V, a vector bundle W on X with a discrete structure group

$$G_W \simeq \pi_1(X) \subset \mathrm{Spin}(10) \subset E_8\,. \tag{2.3}$$

W admits a unique flat connection with holonomy group

$$1 \neq \mathrm{Hol}(W) = \pi_1(X)\,. \tag{2.4}$$

We can then define another vacuum of the $E_8 \times E_8$ heterotic string by considering the vector bundle $V \oplus W$ on X. This preserves $\mathcal{N} = 1$ supersymmetry, but the low energy gauge group is reduced to

$$H_{\mathrm{SU}(4)\times \mathrm{Hol}(W)} \subset \mathrm{Spin}(10)\,. \tag{2.5}$$

We will assume that $\mathrm{Hol}(W)$ is abelian in the following. Then

$$\mathrm{rank}\ H_{\mathrm{SU}(4)\times \mathrm{Hol}(W)} = \mathrm{rank}\ \ \mathrm{Spin}(10) = 5\,. \tag{2.6}$$

[1] More precisely, we only consider "proper" Calabi-Yau threefolds, that is, compact with the full SU(3) holonomy. The fundamental group is then necessarily finite.

Any realistic string vacuum must incorporate the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ at low energies, and this group has rank 4. It follows from eq. (2.6) that, minimally, $H_{SU(4)\times Hol(F)}$ must, in addition to the standard model gauge group, have an extra U(1) gauge factor. It can be shown that a gauged $U(1)_{B-L}$ group, broken at a scale not much larger than the electroweak scale, helps to suppress rapid nucleon decay. Hence, an additional $U(1)_{B-L}$ is a phenomenologically desirable component in the low energy gauge group. Therefore, we would like to choose W so that

$$H_{SU(4)\times Hol(W)} = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}. \tag{2.7}$$

In this section, we will show that this can be arranged provided that

$$Hol(W) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3. \tag{2.8}$$

Other choices of $Hol(W)$ can also lead to eq. (2.7) but, within the context of our construction, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is the smallest group.

We begin by recalling some basic facts about the complexified Lie algebra $\mathfrak{so}(10)_\mathbb{C}$ of Spin(10). First, it is a simple Lie algebra with

$$\dim \mathfrak{so}(10)_\mathbb{C} = 45, \qquad \mathrm{rank}\, \mathfrak{so}(10)_\mathbb{C} = 5. \tag{2.9}$$

Up to conjugation, there exists a unique maximal abelian subalgebra $\mathfrak{h}$, the Cartan subalgebra, with

$$\dim \mathfrak{h} = 5. \tag{2.10}$$

Since $[\mathfrak{h}, \mathfrak{so}(10)_\mathbb{C}] \subset \mathfrak{so}(10)_\mathbb{C}$, the Lie algebra is a module carrying a (fully reducible) linear representation of $\mathfrak{h}$. As $\mathfrak{h}$ is abelian, every irreducible subspace is one-dimensional. This leads to the Cartan-Weyl decomposition of $\mathfrak{so}(10)_\mathbb{C}$ given by

$$\mathfrak{so}(10)_\mathbb{C} = \mathfrak{h} \oplus \sum_\alpha \mathbb{C} e_\alpha. \tag{2.11}$$

Each $\mathbb{C} e_\alpha$ is a one-dimensional $\mathfrak{h}$ module which we can label by some

$$\alpha \in \mathfrak{h}^*, \tag{2.12}$$

where $\mathfrak{h}^*$ denotes the dual space to $\mathfrak{h}$. Each α occurring in eq. (2.11) is called a root of $\mathfrak{so}(10)_\mathbb{C}$, and it follows from eq. (2.9) that $\mathfrak{so}(10)_\mathbb{C}$ has forty roots. Let Φ be this set of roots. They span $\mathfrak{h}^*$, but cannot all be linearly independent. Hence, one can choose a basis $\Delta \subset \Phi$. Actually, Δ can be chosen such that each root $\alpha \in \Phi$ can be written as a $\mathbb{Z}$-linear combination of the roots in Δ with the integral coefficients either all ≥ 0 or all ≤ 0. The elements of Δ are called the simple roots. Since $\dim \mathfrak{h}^* = 5$, there are five simple roots of $\mathfrak{so}(10)_\mathbb{C}$, which we denote by

$$\Delta = \left\{ \alpha^i \,\middle|\, i = 1, \ldots, 5 \right\}. \tag{2.13}$$

To determine the roots explicitly, it is helpful to consider the linear subspace

$$\mathfrak{h}^*_\mathbb{R} = \mathrm{span}_\mathbb{R}\, \Delta. \tag{2.14}$$

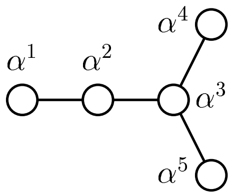

Figure 1: The Spin(10) Dynkin diagram.

Clearly

$$\dim_{\mathbb{R}} \mathfrak{h}^*_{\mathbb{R}} = \dim_{\mathbb{C}} \mathfrak{h}^* = 5\,, \qquad \Phi \subset \mathfrak{h}^*_R \subset \mathfrak{h}^*\,. \tag{2.15}$$

The Killing form on $\mathfrak{so}(10)_{\mathbb{C}}$ is defined by

$$(x,y) = \frac{1}{\lambda}\,\mathrm{Tr}\,\Big(\mathrm{ad}(x)\mathrm{ad}(y)\Big) \qquad \forall x,y \in \mathfrak{so}(10)_{\mathbb{C}}\,, \tag{2.16}$$

where the value of the normalization constant λ will be specified below. Since $\mathfrak{so}(10)_{\mathbb{C}}$ is simple, the Killing form is non-degenerate. This allows us to associate with any $x^* \in \mathfrak{h}^*$ a unique element $x \in \mathfrak{h}$ by

$$x^*(y) = (x,y) \qquad \forall y \in \mathfrak{so}(10)_{\mathbb{C}}\,. \tag{2.17}$$

One can then define a bilinear form $\mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ via

$$\langle x^*, y^* \rangle = (x,y)\,. \tag{2.18}$$

This form and its restriction to $\mathfrak{h}^*_{\mathbb{R}}$ is again non-degenerate. Furthermore, when restricted to $\mathfrak{h}^*_{\mathbb{R}}$, it is positive definite.

The roots of $\mathfrak{so}(10)_{\mathbb{C}}$ are the following. Consider the euclidean space $\mathfrak{h}^*_{\mathbb{R}} \simeq \mathbb{R}^5$ and let e^i, $i = 1,\ldots,5$ be an orthonormal basis. Then the five simple roots are

$$\begin{aligned} \alpha^i &= e^i - e^{i+1}\,, \qquad i = 1,\ldots,4 \\ \alpha^5 &= e^4 + e^5\,. \end{aligned} \tag{2.19}$$

The remaining thirty-five roots can be obtained by acting on eq. (2.19) with Weyl reflections and using the fact that if α is a root, then so is $-\alpha$. There is no need to construct them explicitly. The associated Dynkin diagram is shown in figure 1.

A Chevalley basis of $\mathfrak{so}(10)_{\mathbb{C}}$ consists of five elements h_i, $i = 1,\ldots,5$ which span $\mathfrak{h}$ along with the forty elements e_α, $\alpha \in \Phi - \Delta$. In addition, these satisfy five commutator relations of which we will use the following two:

$$\big[h_i,\ h_j\big] = 0 \tag{2.20a}$$

$$\big[h_i, e_\alpha\big] = \alpha(h_i)e_\alpha\,. \tag{2.20b}$$

We then choose the normalization of the Killing form, eq. (2.16), so that

$$\big(h_i, h_j\big) = \delta_{ij}\,, \qquad \big(e_\alpha, e_\beta\big) = \delta_{\alpha\beta} \tag{2.21}$$

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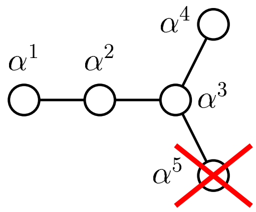

Figure 2: Spin(10) Dynkin diagram with the root α^5 removed.

for all i, j and α, β. We are free to choose each h_i to be the dual element in $\mathfrak{h}$ of $e^i \in \mathfrak{h}^*$. That is

$$e^i\left(h_j\right) = \delta^i_j\,, \qquad i,j = 1,\ldots,5\,. \tag{2.22}$$

For the choice of simple roots in eq. (2.19), it then follows that one must take

$$\lambda = 16\,. \tag{2.23}$$

We will use this normalization for the Killing form henceforth.

2.2 First Wilson line

Let us now find the $\mathfrak{su}(2)_{\mathbb{C}}$ subalgebra associated with the simple root α^5. We denote the generator of its Cartan subalgebra by $H_{(5)}$. It is clear from the marked Dynkin diagram in figure 2 that the product of $H_{(5)}$ with any element of the $\mathfrak{su}(5)_{\mathbb{C}}$ subalgebra associated with the simple roots $\alpha^1,\ldots,\alpha^4$ must vanish. Since each of the $\mathfrak{su}(5)_{\mathbb{C}}$ roots is a linear combination of the four simple roots, it suffices to show that

$$\left[H_{(5)},\ E_{\alpha^i}\right] = 0\,, \qquad i = 1,\ldots,4\,. \tag{2.24}$$

Writing

$$H_{(5)} = \sum_{i=1}^{5} a^i h_i \tag{2.25}$$

and using eqs. (2.19) and (2.22), it follows that all a^i must be identical and, hence,

$$H_{(5)} = a(h_1 + h_2 + h_3 + h_4 + h_5)\,. \tag{2.26}$$

The coefficient a cannot be determined from the Lie algebra alone. To find a, one must construct the one-parameter abelian subgroup of Spin(10) generated by $H_{(5)}$. We denote this subgroup by $\mathrm{U}(1)_{(5)}$. In every representation of Spin(10), $\mathrm{U}(1)_{(5)}$ must be periodic in its parameter θ. This will be the case if and only if it is periodic in the 16-dimensional complex spin representation $\underline{\mathbf{16}}$. The spin representation of the generators h_i, $i = 1,\ldots,5$ can be constructed by standard methods [51]. We find that

$$\left[H_{(5)}\right]_{\underline{\mathbf{16}}} = \frac{a}{2}\left[Y_{(5)}\right]_{\underline{\mathbf{16}}}\,, \tag{2.27}$$

where

$$\left[Y_{(5)}\right]_{\underline{\mathbf{16}}} = \begin{pmatrix} -\mathbb{1}_{10} & & \\ & 3\cdot\mathbb{1}_5 & \\ & & -5 \end{pmatrix} \tag{2.28}$$

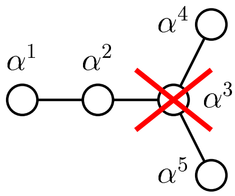

Figure 3: Spin(10) Dynkin diagram with the root α^3 removed.

and $\mathbb{1}_m$ stands for the $m \times m$ unit matrix. An element in the associated $\mathrm{U}(1)_{(5)}$ subgroup is then given by

$$\left[g_{(5)}(\theta)\right]_{\underline{\mathbf{16}}} = \begin{pmatrix} e^{-\frac{ia}{2}\theta}\mathbb{1}_{10} & & \\ & e^{\frac{3ia}{2}\theta}\mathbb{1}_5 & \\ & & e^{-\frac{5ia}{2}\theta} \end{pmatrix}, \tag{2.29}$$

where θ runs over some interval. One can, without loss of generality, assume that $0 \leq \theta < 2\pi$, in which case a must be a multiple of 2. Henceforth, we choose

$$a = 2\,. \tag{2.30}$$

Having determined $H_{(5)}$ and $\mathrm{U}(1)_{(5)}$, we want to consider the finite subgroup of $\mathrm{U}(1)_{(5)}$ generated by the element with $\theta = \frac{2\pi}{3}$. We denote this cyclic subgroup by $(\mathbb{Z}_3)_{(5)}$. This is most easily done by choosing a representation of Spin(10). Since we have already constructed it, we will take this to be the spin representation $\underline{\mathbf{16}}$. It follows from eq. (2.29) that the generator of $(\mathbb{Z}_3)_{(5)}$ is given by

$$\left[g_{(5)}\left(\frac{2\pi}{3}\right)\right]_{\underline{\mathbf{16}}} = \begin{pmatrix} e^{-\frac{2\pi i}{3}}\mathbb{1}_{10} & & \\ & \mathbb{1}_5 & \\ & & e^{-\frac{4\pi i}{3}} \end{pmatrix}. \tag{2.31}$$

Since $e^{-\frac{2\pi i}{3}} \neq 1 \neq e^{-\frac{4\pi i}{3}}$, it follows that the commutant of $(\mathbb{Z}_3)_{(5)}$ in Spin(10) is $\mathrm{SU}(5) \times \mathrm{U}(1)_{(5)}$. Therefore, if we choose a flat line bundle W with holonomy group

$$\mathrm{Hol}(W) = (\mathbb{Z}_3)_{(5)}\,, \tag{2.32}$$

then

$$H_{\mathrm{SU}(4)\times(\mathbb{Z}_3)_{(5)}} = \mathrm{SU}(5) \times \mathrm{U}(1)_{(5)}\,. \tag{2.33}$$

With respect to this subgroup, we can reduce the Spin(10) spin representation as

$$\underline{\mathbf{16}} = (\underline{\mathbf{10}}, -1) \oplus (\underline{\bar{\mathbf{5}}}, 3) \oplus (\underline{\mathbf{1}}, -5)\,. \tag{2.34}$$

2.3 Second Wilson line

Let us now find the Cartan subalgebra of the $\mathfrak{su}(2)_{\mathbb{C}}$ in $\mathfrak{so}(10)_{\mathbb{C}}$ associated with the root α^3. We denote its generator by $H_{(3)}$. It is clear from the marked Dynkin diagram in figure 3 that the product of $H_{(3)}$ with any element of the $\mathfrak{su}(3)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ subalgebra associated with the simple roots α^1, α^2, α^4 and α^5 must vanish. That is

$$\left[H_{(3)},\, E_{\alpha^i}\right] = 0\,, \qquad i = 1,2,4,5\,. \tag{2.35}$$

It follows from our choice of basis, eqs. (2.19) and (2.22), that

$$H_{(3)} = b\,(h_1 + h_2 + h_3) \tag{2.36}$$

for some constant b. Denote by $\mathrm{U}(1)_{(3)}$ the one-parameter subgroup of Spin(10) generated by $H_{(3)}$. We determine the parameter b again by considering the spin representation $\underline{\mathbf{16}}$ of Spin(10) and demanding periodicity in the parameter $0 \leq \theta < 2\pi$ of the $\mathrm{U}(1)_{(3)}$. In a convenient basis

$$[H_{(3)}]_{\underline{\mathbf{16}}} = \frac{b}{2}[Y_{(3)}]_{\underline{\mathbf{16}}}\,, \tag{2.37}$$

where

$$\left[Y_{(3)}\right]_{\underline{\mathbf{16}}} = \begin{pmatrix} \mathbb{1}_6 & & & \\ & -\mathbb{1}_6 & & \\ & & 3\cdot\mathbb{1}_2 & \\ & & & -3\cdot\mathbb{1}_2 \end{pmatrix}. \tag{2.38}$$

A general element of the associated $\mathrm{U}(1)_{(3)}$ subgroup is then given by

$$\left[g_{(3)}(\theta)\right]_{\underline{\mathbf{16}}} = \begin{pmatrix} e^{\frac{ib}{2}\theta}\mathbb{1}_6 & & & \\ & e^{-\frac{ib}{2}\theta}\mathbb{1}_6 & & \\ & & e^{\frac{3ib}{2}\theta}\mathbb{1}_2 & \\ & & & e^{-\frac{3ib}{2}\theta}\mathbb{1}_2 \end{pmatrix}. \tag{2.39}$$

Again, it follows that b must be a multiple of 2, and we will take

$$b = 2\,. \tag{2.40}$$

Having determined $H_{(3)}$ and $\mathrm{U}(1)_{(3)}$, we want to consider the finite subgroup of $\mathrm{U}(1)_{(3)}$ generated by the element with $\theta = 2\pi/3$. We denote this cyclic subgroup by $(\mathbb{Z}_3)_{(3)}$. This is most easily done by choosing a representation of Spin(10), which we take to be $\underline{\mathbf{16}}$. It follows from eq. (2.39) that the generator of $(\mathbb{Z}_3)_{(3)}$ is given by

$$\left[g_{(3)}\left(\frac{2\pi}{3}\right)\right]_{\underline{\mathbf{16}}} = \begin{pmatrix} e^{\frac{2\pi i}{3}}\mathbb{1}_6 & & & \\ & e^{\frac{4\pi i}{3}}\mathbb{1}_6 & & \\ & & \mathbb{1}_2 & \\ & & & \mathbb{1}_2 \end{pmatrix}. \tag{2.41}$$

Since $e^{\frac{2\pi i}{3}} \neq e^{\frac{4\pi i}{3}} \neq 1$, the commutant of $(\mathbb{Z}_3)_{(3)}$ in Spin(10) is $\mathrm{SU}(3)\times\mathrm{SU}(2)^2\times\mathrm{U}(1)_{(3)}$. This is so despite the degeneracy of the two $\mathbb{1}_2$ blocks, as can be verified in several ways. First, note that the decomposition of $\underline{\mathbf{16}}$ with respect to any other subgroup of Spin(10) is inconsistent with eq. (2.41). For example, with respect to $SU(4) \times SU(2)^2$ the $\underline{\mathbf{16}}$ decomposes as $\underline{\mathbf{16}} = (\underline{\mathbf{4}}, \underline{\mathbf{1}}, \underline{\mathbf{2}}) \oplus (\underline{\overline{\mathbf{4}}}, \underline{\mathbf{2}}, \underline{\mathbf{1}})$. Second, one can construct the embedding of the $(\mathbb{Z}_3)_{(3)}$ generator in the $\underline{\mathbf{10}}$ representation of Spin(10). We find that

$$\left[g_{(3)}\left(\frac{2\pi}{3}\right)\right]_{\underline{\mathbf{10}}} = \begin{pmatrix} e^{\frac{4\pi i}{3}}\mathbb{1}_3 & & \\ & e^{\frac{2\pi i}{3}}\mathbb{1}_3 & \\ & & \mathbb{1}_4 \end{pmatrix}, \tag{2.42}$$

which clearly commutes only with the $SU(3) \times SU(2)^2 \times U(1)_{(3)}$ subgroup of Spin(10). Therefore, if we choose a flat line bundle with holonomy group

$$\mathrm{Hol}(W) = (\mathbb{Z}_3)_{(3)}\,, \tag{2.43}$$

then

$$H_{SU(4)\times(\mathbb{Z}_3)_{(3)}} = SU(3) \times \Big(SU(2)\Big)^2 \times U(1)_{(3)}\,. \tag{2.44}$$

With respect to this subgroup

$$\underline{\mathbf{16}} = \Big(\underline{\mathbf{3}}, \underline{\mathbf{2}}, \underline{\mathbf{1}}, 1\Big) \oplus \Big(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, \underline{\mathbf{2}}, -1\Big) \oplus \Big(\underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{2}}, 3\Big) \oplus \Big(\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{1}}, -3\Big)\,. \tag{2.45}$$

Thus far, we have used a basis which most easily allowed us to find the commutant subgroup to $(\mathbb{Z}_3)_{(3)}$. This, however, is not the same basis as was used to write the $(\mathbb{Z}_3)_{(5)}$ generator in eq. (2.31). Changing the basis to the one used in (2.31), we find that

$$\left[g_{(3)}\left(\frac{2\pi}{3}\right)\right]_{\underline{\mathbf{16}}} = \begin{pmatrix} e^{\frac{2\pi i}{3}}\mathbb{1}_6 & & & & & \\ & 1 & & & & \\ & & e^{\frac{4\pi i}{3}}\mathbb{1}_3 & & & \\ & & & \mathbb{1}_2 & & \\ & & & & e^{\frac{4\pi i}{3}}\mathbb{1}_3 & \\ & & & & & 1 \end{pmatrix}. \tag{2.46}$$

2.4 Combined symmetry breaking

Finally, we combine these results and choose a flat rank 2 bundle W with

$$\mathrm{Hol}(W) = (\mathbb{Z}_3)_{(5)} \times (\mathbb{Z}_3)_{(3)}\,. \tag{2.47}$$

It is clear from the $(\mathbb{Z}_3)_{(5)}$ and $(\mathbb{Z}_3)_{(3)}$ generators, expressions (2.31) and (2.46) respectively, that

$$H_{SU(4)\times(\mathbb{Z}_3)_{(5)}\times(\mathbb{Z}_3)_{(3)}} = SU(3) \times SU(2) \times U(1)_{(5)} \times U(1)_{(3)}\,. \tag{2.48}$$

Proceeding sequentially, the $(\mathbb{Z}_3)_{(5)}$ part of $\mathrm{Hol}(W)$ leaves $SU(5) \times U(1)_{(5)}$ invariant and decomposes the representation of Spin(10) as in eq. (2.34). The second factor, $(\mathbb{Z}_3)_{(3)}$, then breaks the SU(5) to $SU(3) \times SU(2) \times U(1)_{(3)}$ and decomposes the $\underline{\mathbf{10}}$, $\underline{\overline{\mathbf{5}}}$ and $\underline{\mathbf{1}}$ of SU(5) as

$$\begin{aligned} \underline{\mathbf{10}} &= \Big(\underline{\mathbf{3}}, \underline{\mathbf{2}}, 1\Big) \oplus \Big(\underline{\mathbf{1}}, \underline{\mathbf{1}}, 3\Big) \oplus \Big(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, -1\Big) \\ \underline{\overline{\mathbf{5}}} &= \Big(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -3\Big) \oplus \Big(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, -1\Big) \\ \underline{\mathbf{1}} &= \Big(\underline{\mathbf{1}}, \underline{\mathbf{1}}, 3\Big)\,. \end{aligned} \tag{2.49}$$

The $\mathrm{SU}(3)\times\mathrm{SU}(2)$ representation content is precisely that of the standard model $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L$. The physical hypercharge and $B-L$ generators are then

$$
\begin{aligned}
[Y] &= \frac{1}{2}[Y_{(5)}] + \frac{5}{6}[Y_{(3)}]\,, \\
[Y_{B-L}] &= \frac{1}{3}[Y_{(3)}]
\end{aligned}
\tag{2.50}
$$

respectively. It follows that the low energy gauge group is

$$
H_{\mathrm{SU}(4)\times(\mathbb{Z}_3)_{(Y)}\times(\mathbb{Z}_3)_{(B-L)}} = \mathrm{SU}(3)_C\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y\times\mathrm{U}(1)_{B-L}\,, \tag{2.51}
$$

as desired.

Equation (2.4) implies that to implement this symmetry breaking pattern one must compactify on Calabi-Yau threefolds X with the property that $(\mathbb{Z}_3)_{(Y)}\times(\mathbb{Z}_3)_{(B-L)}\subseteq\pi_1(X)$. We will consider the minimal case where

$$
\pi_1(X) = (\mathbb{Z}_3)_{(Y)}\times(\mathbb{Z}_3)_{(B-L)}\,. \tag{2.52}
$$

There are examples of such Calabi-Yau threefolds as quotients of toric hypersurfaces. However, in that framework we lack the necessary understanding of stable holomorphic vector bundles. For this reason, we are considering torus fibered Calabi-Yau threefolds with fundamental group eq. (2.52). These have not been presented previously. In the remainder of this paper, we will give an explicit construction of such a manifold and elucidate its properties.

3. Torus fibered Calabi-Yau threefolds

A Calabi-Yau threefold is a compact Kähler manifold (X, J, g) of complex dimension 3 such that $\mathrm{Hol}(g)=\mathrm{SU}(3)$. J is the complex structure and g is the metric, which is then Ricci flat. That is, X is a manifold with a specific complex structure and Kähler metric, and this metric has special holonomy. Such a threefold can be used to compactify heterotic string theory to a $d=4$, $\mathcal{N}=1$ supersymmetric vacuum.

In addition to requiring that X be a Calabi-Yau threefold, we will demand two additional properties:

1. X is torus fibered.[2] That is, there exists an analytic map $\pi : X\to B$, where B is a possibly singular complex surface and the generic fiber is a T^2.

   This is weaker than requiring an elliptic fibration, as a torus fibration does not necessarily have a section $s : B\to X$. In fact, our Calabi-Yau threefold will not admit a section. However, a torus fibration is sufficient to construct stable holomorphic bundles.

2. X has fundamental group $\pi_1(X)=\mathbb{Z}_3\times\mathbb{Z}_3$.

   As discussed in the previous section, such a fundamental group is large enough to break a Spin(10) gauge group via Wilson lines to the standard model group plus $\mathrm{U}(1)_{B-L}$.

[2] This is sometimes called "genus 1 fibered".

There always exists the universal cover $\widetilde{X}$ of X, where, by definition, $\pi_1(\widetilde{X}) = 1$. Pulling back the metric g, we see that $\widetilde{X}$ is a Calabi-Yau manifold if X is. The universal cover $\widetilde{X}$ comes with a free[3] group action of $G \simeq \pi_1(X)$, the deck translations.[4] Conversely, given a simply connected Calabi-Yau threefold $\widetilde{X}$ with a free group action G preserving the metric, then G is a finite group and it can be shown that the quotient $X = \widetilde{X}/G$ is a Calabi-Yau threefold with $\pi_1(X) = G$. So, a non-simply connected Calabi-Yau threefold is equivalent to a simply connected Calabi-Yau threefold with a discrete symmetry.

Therefore, we will construct an elliptically fibered Calabi-Yau threefold $\widetilde{X}$ with automorphism group $G = \mathbb{Z}_3 \times \mathbb{Z}_3$. The base of the elliptic fibration is a complex surface, which we choose (see also [52, 53]) to be a rational elliptic surface dP_9. This is not the only possibility, but is a useful choice since dP_9 is itself elliptically fibered over $\mathbb{P}^1$. Hence, the full Calabi-Yau threefold $\widetilde{X}$ admits a fibration over $\mathbb{P}^1$ such that the generic fiber is the product of two elliptic curves. In fact, such a threefold is automatically the fiber product of two dP_9 surfaces over their base $\mathbb{P}^1$, see [54].

The fiber product is the "universal" way to fill in the pull back diagram

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & B_1 \arrow[d, "\beta_1"'] \\ B_2 \arrow[r, "\beta_2"] & \mathbb{P}^1 \end{tikzcd} \tag{3.1}$$

arising from the projection of the two dP_9 surfaces to a common base $\mathbb{P}^1$. Explicitly, the fiber product is the hypersurface

$$\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2 = \Big\{ (p_1, p_2) \in B_1 \times B_2 \Big| \; \beta_1(p_1) = \beta_2(p_2) \Big\} . \tag{3.2}$$

The fiber product of two dP_9 surfaces is simply connected. To proceed, we must find a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on $\widetilde{X}$. The quotient is then the desired Calabi-Yau threefold X.

It is clear that this group action must preserve the fibration structure. Otherwise the quotient Calabi-Yau manifold would not be torus fibered. Fortunately, this is automatic. Note that the anticanonical class of a dP_9 is the class of the generic fiber F, see [54]. That is,

$$K_{dP_9} = -F . \tag{3.3}$$

Now, any automorphism must preserve the canonical class, and, hence, the fiber class. That is, the image of a generic fiber is an irreducible curve homologous to a fiber. Consequently, it is again a fiber.

To summarize, we want to consider the following:

- A Calabi-Yau threefold $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$, where B_1 and B_2 are two rational elliptic surfaces (dP_9).
- $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ acts on each of B_1 and B_2 such that

[3] A free group action is one without fixed points.

[4] The deck translations are sometimes also called covering automorphisms.

1. The induced action on the base $\mathbb{P}^1$ of B_1 and B_2 is the same. It then follows that the G action extends to $\widetilde{X}$.
2. The G action on $\widetilde{X}$ is free.

- G acts non-trivially on the base $\mathbb{P}^1$. This is not necessary, but yields more interesting actions on the homology. The case of trivial $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on the base $\mathbb{P}^1$ is investigated in [55], Proposition 7.1.

In the following section, we classify such rational elliptic surfaces. It turns out that there exists a one-parameter family of such surfaces. We will explicitly construct this family in section 5.

4. Rational elliptic surfaces with automorphisms

4.1 $\mathbb{Z}_3 \times \mathbb{Z}_3$ actions on rational elliptic surfaces

Let B be a rational elliptic surface, that is a dP_9. First, recall the definition of such a surface. A surface is fibered (over $\mathbb{P}^1$) if there exists a projection map $\beta : B \to \mathbb{P}^1$. A fibered surface is elliptic if the generic fiber is a complex torus and there exists a section, called the 0-section. In the following, we will always denote this section by σ. A rational surface is a blow up of $\mathbb{P}^2$ at a finite number of points. If the surface is also elliptic in addition to being rational, then the number of blowups must be 9. This is the origin of the subscript in "dP_9". Since each blowup increases the Euler characteristic by 1, the rational elliptic surface must have

$$\chi(dP_9) = \chi(\mathbb{P}^2) + 9 = 12\,. \tag{4.1}$$

We can also compute the Euler characteristic from the elliptic fibration point of view. A smooth fibration would be a T^2 bundle and, hence,

$$\chi(T^2 \text{ bundle over } \mathbb{P}^1) = \chi(T^2)\chi(\mathbb{P}^1) = 0 \cdot 2 = 0\,. \tag{4.2}$$

Therefore, the elliptic fibration must degenerate somewhere. The possible singular fibers were classified by Kodaira, and will be reviewed at the beginning of subsection 4.4. However, even without knowing the singular fibers explicitly, we can immediately conclude that

$$\sum_{\text{singular fibers } F_s} \chi(F_s) = \chi(dP_9) = 12\,. \tag{4.3}$$

Using the fact that a rational surface is simply connected, the Euler characteristic determines the single nontrivial (co)homology group

$$H_2(B,\mathbb{Z}) \simeq \mathbb{Z}^{10} \simeq H^2(B,\mathbb{Z})\,. \tag{4.4}$$

Furthermore, let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ act on B such that it maps fibers to fibers. For now, we will describe what one can conclude from this characterization. We demonstrate the existence of such surfaces in section 5. Pick generators g_1, g_2 for G. Abusing notation, also denote the corresponding automorphisms as $g_{1,2} : B \to B$. By our assumptions, one

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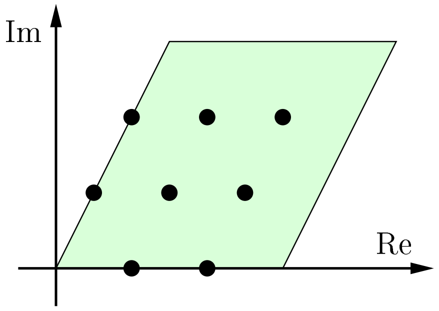

Figure 4: The 8 points on T^2 of order 3.

of them (say, g_1) acts non-trivially on the base $\mathbb{P}^1$. Therefore, we can choose projective coordinates s, t on $\mathbb{P}^1$ such that the induced action $\hat{g}_1$ on the base $\mathbb{P}^1$ is

$$\hat{g}_1 = \beta \circ g_1 \circ \beta^{-1} : \ \mathbb{P}^1 \to \mathbb{P}^1 , \quad [s:t] \mapsto [s:\omega t] \qquad \omega = e^{\frac{2\pi i}{3}} . \tag{4.5}$$

Now, $\mathrm{Aut}(\mathbb{P}^1) = PGL(2,\mathbb{C}) = \mathrm{PSL}(2,\mathbb{C}) = SO^+(3,1)$ does not contain a $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup. The reason is that a finite subgroup of the proper, orthochronous Lorentz group $SO^+(3,1)$ cannot involve a boost and, so, must lie in $\mathrm{SO}(3) \subset SO^+(3,1)$. But SO(3) does not have a $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup as there is no platonic solid with 9 faces.

It follows that the $\mathbb{Z}_3 \times \mathbb{Z}_3$ cannot act faithfully on the base $\mathbb{P}^1$. Thus, some combination of $\hat{g}_1$ and $\hat{g}_2 = \beta \circ g_2 \circ \beta^{-1}$ must act trivially. We will, without loss of generality, assume that

$$\hat{g}_2 = \mathrm{id}_{\mathbb{P}^1} : \ \mathbb{P}^1 \to \mathbb{P}^1 . \tag{4.6}$$

Hence, g_2 leaves each fiber stable. In fact, we can fix the g_2 action uniquely. Pick a generic fiber F. Then $g_2|_F : F \to F$ is some order 3 automorphism of the elliptic curve F. There are 3 possibilities for $g_2|_F$:

1. $g_2|_F = \mathrm{id}_F$, but then g_2 would act trivially on B.

2. F is the hexagonal torus[5] and $g_2|_F$ is multiplication by $e^{\frac{2\pi i}{3}}$. In that case, the complex structure of the fiber $\beta^{-1}(\{\mathrm{pt.}\}) \simeq T^2$ in the elliptic fibration $\beta : B \to \mathbb{P}^1$ must be constant. However, it can be shown that this is impossible.

3. $g_2|_F$ is the translation by a point of order 3, that is, by a point $p \in F \simeq T^2$ satisfying[6] $p \boxplus p \boxplus p = 0$ (but $p \neq 0$ and $p \boxplus p \neq 0$). There are 8 such points, see figure 4.

Everything except translation is ruled out, so g_2 must be the translation by an order 3 section of B, which we will call η. That is,

$$g_2 = t_\eta : \ B \to B . \tag{4.7}$$

[5]Tile the plane by regular hexagons. Then the vertices and midpoints form the hexagonal lattice. The quotient of the complex plane by the hexagonal lattice is the hexagonal torus. It is the only elliptic curve with a $\mathbb{Z}_3$ subgroup in the automorphism group.

[6]Here, by $\boxplus$ we denote the addition of points on a torus.

Now that we have determined the form of the g_2 action, let us proceed to investigate g_1. There is another section defined by the G action on B, that is, the image of the 0-section σ under g_1. Denote this section by

$$\xi = g_1(\sigma)\,. \tag{4.8}$$

We can think of the g_1 action as first applying $t_{-\xi} \circ g_1$ (which fixes the 0-section) and then translating by ξ,

$$g_1 = t_\xi \circ \Big(t_{-\xi} \circ g_1\Big)\,. \tag{4.9}$$

In other words, we can factor the automorphism g_1 into t_ξ, the translation by ξ, and another automorphism α_B which leaves the 0-section invariant. That is

$$g_1 = t_\xi \circ \alpha_B : \; B \to B \tag{4.10}$$

where

$$\alpha_B = t_{-\xi} \circ g_1 \tag{4.11}$$

and $\alpha_B(\sigma) = \sigma$.

4.2 The Mordell-Weil group

The requirement that there be additional sections, besides σ, restricts B considerably. Let us review a few facts about sections. First of all, a generic fiber of B is a smooth elliptic curve. Because we insist that B has a section σ, there is a marked point on the elliptic curve where σ intersects the fiber. This is precisely what is needed to define a group law on the curve. One can think of an elliptic curve as a quotient $\mathbb{C}/\Lambda$ of the complex plane and a lattice. The 0-section fixes the origin of the complex plane. The group law is simply the addition of complex numbers, modulo the lattice Λ.

Hence, one can add sections to get another section just by adding the points on each fiber. We will denote this addition of sections by "$\boxplus$", the same symbol as for the addition of points on a single torus, to distinguish it from the addition of classes in $H_2(B,\mathbb{Z})$ which is denoted by "$+$". All sections form a countable set, see [56], so they compose a discrete abelian group under "$\boxplus$" called the Mordell-Weil group $E(K)$.

Any section is, of course, a 2-dimensional submanifold and, hence, defines a class in $H_2(B,\mathbb{Z})$. In fact, the section is uniquely determined by its homology class. So we can think of $E(K)$ as a subset of $H_2(B,\mathbb{Z})$. However, it turns out that $E(K)$ can not, in general, be a subgroup.[7] So our only hope to describe the Mordell-Weil group in terms of the homology group is to write $E(K)$ as a quotient of $H_2(B,\mathbb{Z})$. In fact, this is possible. There is a subgroup $T \subset H_2(B,\mathbb{Z})$ such that

$$E(K) \simeq H_2(B,\mathbb{Z})/T\,, \tag{4.12}$$

as is proven in [56]. Moreover, the subgroup T has a rather simple description. T is the subgroup generated by the homology classes of

[7] To see that it is not a subgroup, note that for some rational elliptic surfaces $E(K)$ contains torsion elements while $H_2(B,\mathbb{Z})$ is always torsion free.

1. the 0-section σ,
2. a generic fiber F, and
3. the irreducible components of singular fibers not intersecting σ. Note that the non-singular fibers have only one irreducible component which must therefore intersect σ.

The most natural description of the Mordell-Weil group is, then, in terms of the short exact sequence

$$0 \longrightarrow T \longrightarrow H_2(B,\mathbb{Z}) \longrightarrow E(K) \longrightarrow 0\,, \tag{4.13}$$

which encodes T as a subgroup and $E(K)$ as the corresponding quotient of $H_2(B,\mathbb{Z})$.

4.3 The Mordell-Weil lattice

There is more structure in the Mordell-Weil group than just the addition law. The intersection pairing in $H_2(B,\mathbb{Z})$ induces the so-called height pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle:\; E(K)\times E(K)\to\mathbb{Q} \tag{4.14}$$

in the following way. Project the homology classes of the sections to the orthogonal complement $T^\perp\subset H_2(B,\mathbb{Q})$, and take their intersection there. It is, in general, a rational number since the projection does not always end up in $H_2(B,\mathbb{Z})$. The Mordell-Weil group together with the (nondegenerate) height pairing forms the Mordell-Weil lattice.

We have the following explicit formula (see [56]) for the height pairing of any two sections μ, ν. It is

$$\langle\mu,\nu\rangle = 1+\mu\sigma+\nu\sigma-\mu\nu-\sum_{s\in\mathbb{P}^1}\mathrm{contr}_s(\mu,\nu)\,, \tag{4.15}$$

where multiplication of sections denotes the intersection product of their homology classes and $\mathrm{contr}_s(\mu,\nu)$ is the entry of the inverse of the intersection matrix T_s associated with the irreducible components not intersecting σ. Note that if the fiber $\beta^{-1}(s)$ is smooth, then T_s is a 0×0 matrix. Therefore, the sum in eq. (4.15) really only runs over the points $s\in\mathbb{P}^1$ where $\beta^{-1}(s)$ is a singular fiber.

For example, assume that $\beta^{-1}(s)$ is an I_3 singular fiber of the elliptic fibration. The I_3 singular fiber consists of three irreducible components, intersecting in a triangle. The 0-section σ intersects one of the components, the so-called neutral component. Thus, we are left with two components not intersecting the 0-section, see figure 5. Their intersection matrix is the A_2 Cartan matrix with inverse

$$\begin{pmatrix}2&-1\\-1&2\end{pmatrix}^{-1}=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}\end{pmatrix}. \tag{4.16}$$

Therefore, in this case, $\mathrm{contr}_s(\mu,\nu)$ can take the following three values:

- $\mathrm{contr}_s(\mu,\nu)=0$ if either μ or ν intersect the neutral component.

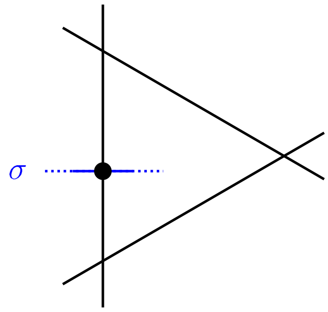

Figure 5: Singular fiber of the I_3 type.

Type of $\beta^{-1}(s)$	I_1	$I_m, m \geq 1$	II	III	IV	I_m^*	II^*	III^*	IV^*
T_s	$()$	A_{m-1}	$()$	A_1	A_2	D_{m+4}	E_8	E_7	E_6
Euler characteristic	1	m	2	3	4	$m+6$	10	9	8

Table 1: Kodaira singular fibers.

- $\mathrm{contr}_s(\mu,\nu) = \frac{2}{3}$ if both μ and ν intersect the same component of the I_3 fiber and not the neutral component.
- $\mathrm{contr}_s(\mu,\nu) = \frac{1}{3}$ if μ, ν and σ each intersect a different component of the I_3 fiber.

It turns out that requiring 3-torsion[8] in the Mordell-Weil group is highly restrictive. Very few combinations of singular fibers are possible, as will be discussed in the following section.

4.4 Configurations of singular fibers

Let us recall Kodaira's classification of singular fibers. If $\beta^{-1}(s)$ is reducible, then the irreducible components not intersecting the 0-section form a root lattice (a sublattice of $H_2(B,\mathbb{Z})$) of the A-D-E type,[9] which we will denote by T_s. If $\beta^{-1}(s)$ is irreducible, then we write $T_s = ()$, the empty lattice. All possible singular fibers are listed in table 1. Note that a lattice T_s either uniquely determines the Kodaira type of the singular fiber, or is associated with a small number of possible fibers. We can use this to find all possible configurations of singular fibers that can occur when $E(K)$ has 3-torsion.

For any point $s \in \mathbb{P}^1$, we can decompose the fiber $\beta^{-1}(s)$ into irreducible components. These components determine homology classes. Therefore, we have the embeddings

$$\forall\ s \in \mathbb{P}^1 :\ T_s \subset H_2(B,\mathbb{Z})\,. \tag{4.17}$$

Of course, if $\beta^{-1}(s)$ is smooth, then the empty lattice $T_s = ()$ is embedded trivially. Since the fibers over different points do not intersect, all T_s must be embedded together as mutually orthogonal sublattices

$$\bigoplus_{s\in\mathbb{P}^1} T_s \subset H_2(B,\mathbb{Z})\,. \tag{4.18}$$

[8]Throughout this paper we use "torsion" in the group theoretical sense, that is, group elements of finite order: $g^n = 1$ for some $n \in \mathbb{Z}$.

[9]The A-D-E lattices are the root lattices of the A-D-E Lie groups.

No.	39	51	61	63	66	68	69
T	$A_2^{\oplus 3}$	$A_5 \oplus A_2$	$A_2^{\oplus 3} \oplus A_1$	A_8	$A_5 \oplus A_2 \oplus A_1$	$A_2^{\oplus 4}$	$E_6 \oplus A_2$
$E(K)$	$A_2^* \oplus \mathbb{Z}_3$	$A_1^* \oplus \mathbb{Z}_3$	$\langle \frac{1}{6} \rangle \oplus \mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_6$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3$

Table 2: Possible Mordell-Weil groups with 3-torsion. In the first row we have also listed the case number of [57]. $\langle 1/6 \rangle$ denotes $\mathbb{Z}$ with the inner product matrix $(1/6) \in \mathrm{Mat}(1, \mathbb{Q})$.

No.	$\oplus T_s$	$E(K)$	$\beta^{-1}([1:0])$	$\beta^{-1}([0:1])$	Other sing. fibers
39a	$A_2^{\oplus 3}$	$A_2^* \oplus \mathbb{Z}_3$			$3I_3$, $3I_1$
(39b)	$A_2^{\oplus 3}$	$A_2^* \oplus \mathbb{Z}_3$	II	I_1	$3I_3$
39c	$A_2^{\oplus 3}$	$A_2^* \oplus \mathbb{Z}_3$			$3IV$
51	$A_5 \oplus A_2$	$A_1^* \oplus \mathbb{Z}_3$	I_6	I_3	$3I_1$
61a	$A_2^{\oplus 3} \oplus A_1$	$\langle \frac{1}{6} \rangle \oplus \mathbb{Z}_3$	I_2	I_1	$3I_3$
(61b)	$A_2^{\oplus 3} \oplus A_1$	$\langle \frac{1}{6} \rangle \oplus \mathbb{Z}_3$	III		$3I_3$
63	A_8	$\mathbb{Z}_3$	I_9		$3I_1$
68	$A_2^{\oplus 4}$	$\mathbb{Z}_3^2$	I_3		$3I_3$
69	$E_6 \oplus A_2$	$\mathbb{Z}_3$	IV^*	IV	

Table 3: Possible Configurations of Singular Fibers.

The embeddings of sums of A-D-E root lattices T_s in the $H_2(B, \mathbb{Z})$ lattice are completely classified in [57]. Here, we only quote their result. There are 74 distinct sums

$$\bigoplus_{s \in \mathbb{P}^1} T_s = T \subset H_2(B, \mathbb{Z}) \,. \tag{4.19}$$

For each of these, one can then determine the Mordell-Weil group from eq. (4.13). This is a classical result which can also be found in [57]. It turns out that only the 7 lattice embeddings given in table 2 lead to 3-torsion.

It remains to list all possible configurations of singular fibers corresponding to the 7 possible lattices T. Each singular fiber either is or is not one of the g_1-stable fibers. In the case where the fiber is g_1-stable, it sits either over $[1:0]$ or $[0:1] \in \mathbb{P}^1$. Of course, nothing changes if we switch these two points, so we can require that $\chi(\beta^{-1}([1:0])) \geq \chi(\beta^{-1}([0:1]))$ without loss of generality. In the second case where the fiber is not g_1-stable, the same singular fiber occurs 3 times, cyclically permuted by g_1.

We have found all consistent lattice embeddings. However, this is not enough to infer the singular fibers. For example, we cannot determine how many I_1 fibers there are, since this singular fiber does not contribute to the T lattice. The additional criterion we will use is that the Euler characteristic of the singular fibers must add up to 12, see eq. (4.3). This fixes the singular fibers uniquely except in case number 39 and 61 for which there are three (39a,b,c) and two (61a,b) combinations respectively. We list[10] these combinations in table 3.

[10]The attentive reader will notice that case 66 is missing. This is ruled out since $\oplus T_s = A_5 \oplus A_2 \oplus A_1$ cannot come from a configuration of singular fibers. The three corresponding singular fibers cannot all sit over the two special points $[1:0]$, $[0:1] \in \mathbb{P}^1$.

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No.	39a	39c	51	61a	63	68	69
Number of moduli	1	0	0	0	0	0	0

Table 4: Moduli for the surfaces with 3-torsion in the Mordell-Weil group.

So far we only checked that the configuration of singular fibers gives a consistent lattice embedding in $H_2(B,\mathbb{Z})$ and has the right Euler characteristic. However, not all combinations of singular fibers (with Euler characteristic adding up to 12) can occur in a rational elliptic surface. Picking the singular fibers amounts to choosing the monodromies of the smooth fibers as one goes around the singular fiber. But not all choices of monodromy matrices are allowed since encircling all singular fibers is a contractible loop. Hence, there is a condition that the product of all monodromies is 1. In fact, by comparing with the list of actually realized singular fibers (see [58]), we can exclude cases 39b and 61b as well. This is why they appear in brackets in table 3.

Finally, let us consider the dimension of the moduli space for these rational elliptic surfaces. A generic dP_9 has 12 singular fibers and is uniquely determined by their position. Choosing coordinates on the base $\mathbb{P}^1$ amounts to fixing 3 points. Hence, the moduli space is $12 - 3 = 9$ dimensional. Of course, we are dealing with highly non-generic dP_9 surfaces which admit $\mathbb{Z}_3 \times \mathbb{Z}_3$ automorphisms. By choosing g_1 as in eq. (4.5), we have already designated 2 points to be the two fixed points. So the position of one triple of singular fibers is the only remaining freedom in the choice of the coordinate system. We see that the surface in case 39a has one complex modulus, which is the position of the second triple of singular fibers. All remaining surfaces are isolated. The number of parameters for each case is tabulated in table 4.

Furthermore, cases 39c, 63, and 68 are special points in the moduli space of 39a. At these points, respectively, I_1 and I_3, $3I_3$, and $3I_1$ fibers collide. This can be seen explicitly as a limit of the Weierstrass model, see appendix A.

4.5 Actions with restricted fixed point sets

Our goal is to construct free $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ group actions on the fiber product $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$. We will show that not all of the dP_9 surfaces in table 3 can admit such a group action. Rather, there is an additional restriction. Obviously, the fiber over $s \in \mathbb{P}^1$ contains a $\mathbb{Z}_3 \subset G$ fixed point if the fibers $\beta_1^{-1}(s) \subset B_1$ and $\beta_2^{-1}(s) \subset B_2$ simultaneously have a $\mathbb{Z}_3$ fixed point. Conversely, if there are no fixed points for all choices of $\mathbb{Z}_3 \subset G$ then the action is free.[11]

First, let us review a standard argument showing that there must be at least one fixed point on each rational elliptic surface B. This shows that we will always have points $s \in \mathbb{P}^1$ such that either $\beta_1^{-1}(s)$ or $\beta_2^{-1}(s)$ contains a fixed point. The argument is as follows. For any fiber preserving automorphism $1 \neq K \in \mathrm{Aut}(B)$, the resolution[12] $\widehat{B/K}$ must again be a rational elliptic surface. This is a well known consequence of the classification of

[11] $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ is generated by g_1 and g_2. Therefore we have to check that g_1, g_2, and $g_1 g_2$ do not have fixed points.

[12] Blow up of the orbifold singularities.

surfaces. However, since it is an important part of our argument, we review the proof it in appendix B. Now assume K acts without fixed points on B. Then, the K quotient would be smooth and

$$12 = \chi\left(\widehat{B/K}\right) = \chi(B/K) = \frac{12}{|K|} < 12\,. \tag{4.20}$$

This is a contradiction. Therefore there must be a fixed point of K in B.

Consider the subgroup

$$G_1 = \{1, g_1, g_1^2\}\,. \tag{4.21}$$

By construction, this acts non-trivially on the base $\mathbb{P}^1$. Of course, the G_1 fixed points must then lie in the stable fibers

$$F_0 = \beta^{-1}([1:0])\,, \qquad F_\infty = \beta^{-1}([0:1])\,. \tag{4.22}$$

A simple application of the previous argument proves that there must be a G_1 fixed point in either F_0 or F_∞ in every case. The same holds for the subgroup generated by $g_1 g_2$, which also acts non-trivially on the base $\mathbb{P}^1$. Finally, consider the subgroup

$$G_2 = \{1, g_2, g_2^2\}\,. \tag{4.23}$$

Since g_2 is a non-zero translation, there are no G_2 fixed points on smooth fibers. Counting the Euler characteristic of the resolution, one can determine which singular fibers contain G_2 fixed points. For example, in case 39a there is a G_2 fixed point in each of the three I_1 fibers.

After these preliminaries, we can now discuss which dP_9 surfaces can be used to construct fiber products $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$ with fixed point free G_1 actions. It turns out that the G_2 action can always be arranged to be fixed point free on the fiber product, which is why we concentrate on G_1 in the following. We naturally must distinguish between fixed points over $[1:0], [0:1] \in \mathbb{P}^1$ and the remaining fibers.

First, if $s \neq [1:0], [0:1] \in \mathbb{P}^1$, then we can always rescale (that is, act by the commutant of $\hat{g}_1$ in $PGL(2)$) the base $\mathbb{P}^1$ in, say, B_2 so that the singular fibers in B_1 and B_2 are not paired up in the fiber product. It follows that these fixed points cannot prevent us from constructing a free action.

The only problematic case is for $s = [1:0]$ or $[0:1]$, since these points are fixed by the G_1 action. We cannot rotate the relative position of these points in B_2, because then the G_1 action would no longer extend to the fiber product $\widetilde{X}$. Furthermore, we have already seen that there must be a G_1 fixed point in at least one of the two stable fibers F_0, F_∞. This fiber must not be paired in the fiber product with another fiber containing a G_1 fixed point. Therefore, we can not use any surface that has a fixed point in both F_0 and F_∞. If the fibers are smooth tori, this poses no additional restriction. As we have seen in figure 4, $\mathbb{Z}_3$ can act freely on T^2. Hence, it is important to know which singular fibers allow free $\mathbb{Z}_3$ actions. It turns out to be quite difficult for G_1 to act freely on the singular fibers, as we now discuss.

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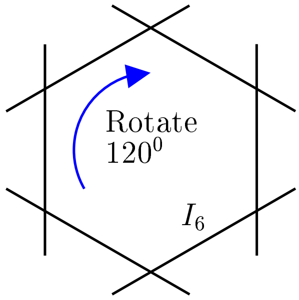

Figure 6: An automorphism of the I_6 singular fiber.

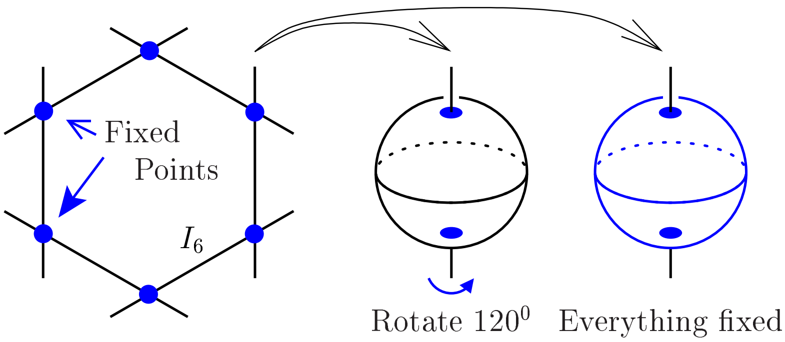

Figure 7: Other automorphisms of I_6. The 2 possible actions on the irreducible components ($\mathbb{P}^1$) are indicated.

An obvious necessary condition for a Kodaira fiber to allow a free $\mathbb{Z}_3$ action is that the Euler characteristic is divisible by 3. This leaves only I_{3m}, III, I^*_{3m} and III^*. Therefore, we can immediately rule out cases 61a and 69 since, in these cases, neither of F_0, F_∞ has suitable Euler characteristic.

It is more difficult to exclude case 51. This will occupy the remainder of this section. Any reader not interested in the details is encouraged to jump to the summary at the end of this section. In case 51, the two fibers in question, $F_0 = I_6$ and $F_\infty = I_3$, certainly do have free $\mathbb{Z}_3$ automorphisms. Even so, the overall topology of the rational elliptic surface B prevents one from having these special free automorphisms induced at F_0 and F_∞. The reason is again that for any fiber preserving automorphism $G \in \mathrm{Aut}(B)$, the resolution $\widehat{B/G}$ must again be a rational elliptic surface. What do the possible $\mathbb{Z}_3$ automorphisms of F_0 and F_∞ look like? As an example, consider $F_0 = I_6$. One possible automorphism is the rotation of the hexagon, as depicted in figure 6. This is obviously a free $\mathbb{Z}_3$ action. Another possible automorphism is shown in figure 7. Here we act on each irreducible component separately. In this case, there are obviously at least 6 fixed points. The group of all possible automorphisms, $\mathrm{Aut}(I_6)$, is then a semidirect product of these two types.

Now, assume that we are in case 51 and the only G_1 fixed points are in the fiber F_0. There must be a fixed point, so $g_1|_{F_0}$ cannot be the automorphism of the I_6 depicted in figure 6. This means that $g_1|_{F_0} : F_0 \to F_0$ must fix at least the 6 intersection points of the irreducible components. Depending on the action on the components themselves (see figure 7), the intersection points may or may not become orbifold singularities. However,

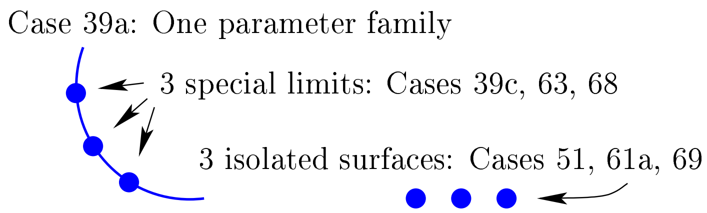

Figure 8: Hiker's guide to dP_9 with $\mathbb{Z}_3 \times \mathbb{Z}_3$ action.

in either case, the fiber in the resolution $\widehat{B/G_1}$ must be an I_n, $n \geq 6$. We can determine n by a simple Euler characteristic computation.

$$12 = \chi\Big(\widehat{B/G_1}\Big) = \chi(I_n) + \frac{1}{3}\Big(3\chi(I_1) + \chi(I_3)\Big) \quad \Rightarrow \quad n = 10\,. \tag{4.24}$$

This is a contradiction, since I_{10} cannot appear as a singular fiber in a rational elliptic surface. The rank 9 lattice generated by all the irreducible components except one, added to the lattice generated by the 0-section and the fiber, has overall rank 11. Therefore, it is too large to be embedded in the homology lattice $H_2(B, \mathbb{Z}) \simeq \mathbb{Z}^{10}$. It remains to check that the other distribution of fixed points in case 51, with all fixed points in F_∞, also can not occur. By the same argument as above, the resolution $\widehat{B/G_1}$ has singular fibers I_9, I_2 and I_1. This is again excluded on dimensional grounds, since there are again 9 irreducible components (in I_9 and I_2) not intersecting the 0-section.

To summarize, we have found a classification of rational elliptic surfaces with fiberwise $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ automorphisms which act non-trivial on the base, see figure 8. There is a single 1-parameter family of such surfaces. These surfaces are the only ones that can be used to construct a free G action on the fiber product $\widetilde{X}$. The three additional isolated surfaces will not lead to fixed point free actions on the fiber product. Therefore, in the remainder of the paper, we will focus on the generic case 39a.

5. Explicit realizations

5.1 Weierstrass generalities

5.1.1 The Weierstrass equation

Any elliptic surface $\beta : B \to \mathbb{P}^1$ can be encoded in a Weierstrass polynomial

$$y^2 z = x^3 + a(t) x z^2 + b(t) z^3\,, \tag{5.1}$$

where x, y, z are homogeneous coordinates

$$[x : y : z] = [\lambda x : \lambda y : \lambda z] \qquad \forall \lambda \neq 0 \tag{5.2}$$

and t is an affine coordinate on the base $\mathbb{P}^1$. For a rational elliptic surface, $a(t)$ and $b(t)$ are polynomials in t of degree 4 and 6. Implicit in the Weierstrass equation, there are the following conventions relating to the other coordinate chart $t \to \overline{t} = 1/t$.

$$x \to \overline{x} = \frac{1}{t^2} x \,, \qquad y \to \overline{y} = \frac{1}{t^3} y \,, \qquad z \to \overline{z} = z \,, \tag{5.3}$$

$$a \to \overline{a} = \frac{1}{t^4} a \,, \qquad b \to \overline{b} = \frac{1}{t^6} b \,. \tag{5.4}$$

These rules are chosen so that after the coordinate change, one still has a Weierstrass polynomial of the form eq. (5.1).

These non-trivial transformation rules mean that a and b are not functions but, rather, sections of the line bundles

$$a \in \Gamma\Big(\mathcal{O}_{\mathbb{P}^1}(4)\Big) \,, \qquad b \in \Gamma\Big(\mathcal{O}_{\mathbb{P}^1}(6)\Big) \,, \tag{5.5}$$

where $\mathcal{O}(n)$ is the sheaf of analytic functions homogeneous of degree n. Similarly, $[x : y : z]$ is a section of the following bundle P of projective spaces over $\mathbb{P}^1$:

$$\begin{array}{ccc} \mathbb{P}^2 \longrightarrow & P = \mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}\Big) & . \\ & \Big\downarrow p & \\ & \mathbb{P}^1 & \end{array} \tag{5.6}$$

The Weierstrass equation is then well defined on P, and we denote its solution set by

$$W_B = \Big\{ ([x : y : z], t) \in P \;\Big|\; \text{Weierstrass equation is satisfied} \Big\} \,. \tag{5.7}$$

A fundamental fact is that, although they encode the same information, in general

$$B \neq W_B \,. \tag{5.8}$$

Note that a fiber of W_B is a curve in the corresponding fiber of P, that is, a cubic in $\mathbb{P}^2$. But a cubic in $\mathbb{P}^2$ has at most 3 irreducible components, whereas the singular fibers of B may contain up to 9 irreducible components. Furthermore, W_B is, in general, a singular variety while B is smooth.

This suggests that we have to resolve the singularities to identify these spaces. Indeed,

$$B = \widehat{W_B} \,. \tag{5.9}$$

The advantage of the Weierstrass model is that it is very convenient for computations. For example, it is easy to read off the singular fibers of B directly from the Weierstrass equation. The fibration degenerates whenever the discriminant

$$D(t) = 4a(t)^3 + 27b(t)^2 \tag{5.10}$$

vanishes. Since $D(t)$ is a polynomial of degree 12 in t, it follows that there are at most 12 singular fibers. The precise nature of a singular fiber can then be read off from the order of vanishing of D, a and b at that fiber. The requisite information is presented in table 5, see [59].

Kodaira fiber	order$\big(a(t-t_0)\big)$	order$\big(b(t-t_0)\big)$	order$\big(D(t-t_0)\big)$
I_0	any	any	0
I_0	≥ 3	3	6
I_0	2	≥ 4	6
I_n, $n \geq 1$	0	0	n
I_n, $n \geq 1$	2	3	$n+6$
I_0^*	2	3	6
II	≥ 1	1	2
IV^*	≥ 3	4	8
III	1	≥ 2	3
III^*	3	≥ 5	9
IV	≥ 2	2	4
II^*	≥ 4	5	10

Table 5: Singular fibers of the Weierstrass equation at $t = t_0$.

5.1.2 Sections

Another advantage of the Weierstrass model is that one can describe sections explicitly. An arbitrary section can be written as

$$\rho : \mathbb{P}^1 \to W_B\,, \qquad t \mapsto [\rho_x(t) : \rho_y(t) : \rho_z(t)]\,. \tag{5.11}$$

Hence, it is defined by three polynomials ρ_x, ρ_y, and ρ_z satisfying the Weierstrass equation. Of course, globally, ρ_x, ρ_y, and ρ_z are sections of some sheaves. Therefore, they transform like x, y, and z as we change the coordinate patch. As long as ρ_z is not identically zero, one can use the homogeneous rescaling and write the section as

$$\rho : \mathbb{P}^1 \to W_B\,, \quad t \mapsto [\tilde{\rho}_x(t) : \tilde{\rho}_y(t) : 1]\,, \tag{5.12}$$

where $\tilde{\rho}_x$ and $\tilde{\rho}_y$ are now locally meromorphic functions. Globally, they must be sections of the sheaf of meromorphic functions of homogeneous degree 2 and 3. That is,

$$\tilde{\rho}_x \in \Gamma\Big(\mathfrak{M}(2)\Big)\,, \qquad \tilde{\rho}_y \in \Gamma\Big(\mathfrak{M}(3)\Big)\,. \tag{5.13}$$

In all cases, the section

$$\sigma : \mathbb{P}^1 \to B\,, \qquad t \mapsto [0 : 1 : 0] \tag{5.14}$$

always exist and is, by convention, declared to be the 0-section.

Similarly, the addition law has a clear geometric meaning. Recall that to define the addition law for sections, one need only specify the addition of points in each fiber. By continuity, we only have to do this for the generic (smooth) fibers. In the Weierstrass model, these are simply smooth cubic curves in $\mathbb{P}^2$. Explicitly, fix $t = t_0$ and define the curve

$$C = \Big\{ [x : y : z] \;\Big|\; y^2 z = x^3 + a(t_0) x z^2 + b(t_0) z^3 \Big\} \subset \mathbb{P}^2\,. \tag{5.15}$$

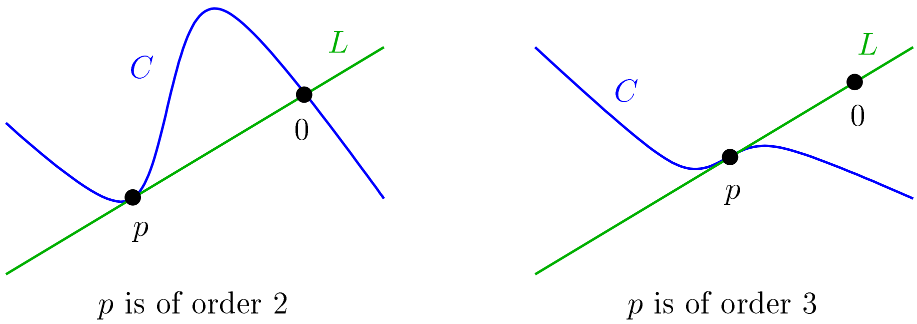

Figure 9: Special cases of the geometric group law.

Together with the chosen origin

$$0 = [0 : 1 : 0] \in C\,, \tag{5.16}$$

this is an abelian group.

To define the group law, we have to specify the sum $p_1 \boxplus p_2$ for any two points $p_1, p_2 \in C$. However, for our purposes, it is more useful to give the group law in a more symmetric way. For any two points p_1 and p_2, one must specify a unique p_3 such that

$$p_1 \boxplus p_2 \boxplus p_3 = 0\,. \tag{5.17}$$

The group law is then as follows. Any line

$$L = \Big\{ [x : y : z] \;\Big|\; \ell_1 x + \ell_2 y + \ell_3 z = 0 \Big\} \subset \mathbb{P}^2\,, \quad (\ell_1, \ell_2, \ell_3) \in \mathbb{C}^3 - \{0\} \tag{5.18}$$

intersects the cubic C in three points

$$\Big\{ p_1,\, p_2,\, p_3 \Big\} = C \cap L\,, \tag{5.19}$$

possibly with multiplicity. By definition, these points add up to zero,

$$p_1 \boxplus p_2 \boxplus p_3 = 0\,. \tag{5.20}$$

It is clear that p_3 is unique, since there is only a single line L passing through p_1 and p_2. One can geometrically check that this addition law satisfies the group axioms.

Because we allow for multiplicities, that is, L may be tangent to C, there are the following two important special cases (see figure 9). First, $p \in C$ is of order 2 if $p \boxplus p \boxplus 0 = 0$. The line L is then tangent to C at p and intersects C transversely at 0. Second, if p is a point of order 3 then $p \boxplus p \boxplus p = 0$. Geometrically, this means that L intersects C in a flex. A smooth cubic always has 9 flexes, one being $[0 : 1 : 0]$ in the Weierstrass form.

5.2 Weierstrass model

Although we know from the above classification that surfaces with suitable singular fibers and a 3-torsion section exist, it is not obvious that they actually have a $G = \mathbb{Z}_3 \times \mathbb{Z}_3$

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action. One way to ensure this is to write down an explicit realization. Consider the following Weierstrass equation

$$y^2z = x^3 + \left[t + t^4\left(\gamma + \frac{1}{48}\right)\right] xz^2 + \left[1 + 2\gamma t^3 + \left(\gamma^2 - \frac{1}{1728}\right)t^6\right] z^3\,, \tag{5.21}$$

where t is the coordinate on the base $\mathbb{P}^1$ and γ is the one complex parameter of the family. This turns out to correspond to case 39a in table 3. First, we check that it has the correct configuration of singular fibers. The discriminant is

$$D = 27\left[1 + \left(\gamma + \frac{1}{24}\right)t^3\right]^3 \left[1 + \left(\gamma + \frac{5}{216}\right)t^3\right]. \tag{5.22}$$

From this factorization and table 5, we can easily read off the singular fibers. For generic γ, there is an I_3 at each of the three roots of $t^3 = -(\gamma + \frac{1}{24})^{-1}$ and an I_1 at each of the three roots of $t^3 = -(\gamma + \frac{5}{216})^{-1}$.

The generators of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ are $g_1 = t_\xi \circ \alpha_B$ and $g_2 = t_\eta$, so we have to describe ξ, η and α_B. First, α_B is the order 3 automorphism

$$\alpha_B : B \to B\,, \quad \big([x:y:z],\, t\big) \mapsto \big([\omega x : y : z],\, \omega^{-1}t\big) \qquad \omega = e^{\frac{2\pi i}{3}}\,. \tag{5.23}$$

It obviously preserves the Weierstrass equation eq. (5.21) as well as the 0-section, eq. (5.14). The order 3 section is

$$\eta : \mathbb{P}^1 \to B\,, \quad t \mapsto \left[\frac{1}{12}t^2 : 1 + \left(\gamma + \frac{1}{24}\right)t^3 : 1\right]. \tag{5.24}$$

It is straightforward to check that η is of order 3, that is

$$\eta \boxplus \eta \boxplus \eta = \sigma\,, \tag{5.25}$$

or, equivalently, $\eta(t)$ is a flex for all t. Furthermore, η is preserved by α_B. Hence, g_1 and g_2 commute.

Finally, we should give the section ξ satisfying

$$\Big(t_\xi \circ \alpha_B\Big)^3 = 1 \quad \Leftrightarrow \quad \alpha_B^2\xi \boxplus \alpha_B\xi \boxplus \xi = 0\,. \tag{5.26}$$

Any general equation for the section ξ is exceedingly complicated since the section must have monodromies around the special values for γ. Therefore, the coefficients of the polynomials $\xi_x(t)$, $\xi_y(t)$ must contain roots of polynomials in γ. To simplify matters, we will choose a specific value $\gamma = -\frac{1}{48}$. Note that this value is not one of the special points corresponding to cases 39c, 63, 68. However, it simplifies the Weierstrass equation somewhat, which now becomes

$$y^2z = x^3 + txz^2 + \left[1 - \frac{1}{24}t^3 - \frac{1}{6912}t^6\right] z^3\,. \tag{5.27}$$

For this special value we can now write down the section ξ. Pick one root of $r^3 = \frac{\sqrt{3}}{96}i$. Then, the following is a section

$$\xi : \mathbb{P}^1 \to B\,, \quad t \mapsto \left[\frac{\sqrt{3}}{6}i\left(4rt + \frac{1}{r}\right) : \frac{1}{\sqrt{3}}i\left(1 + \frac{1}{48}t^3\right) : 1\right]. \tag{5.28}$$

One immediately recognizes that the α_B-action on ξ corresponds precisely to the choice of the 3rd root for r. Moreover, since only one coordinate depends on r, the three α_B-images lie on a line. That is, the points add to 0 on each fiber. Hence eq. (5.26) is satisfied. For other nondegenerate values of γ, one can also find sections since the Mordell-Weil group must remain the same. Furthermore, since the α_B-action on the discrete Mordell-Weil group has to be invariant under smooth deformations, the section likewise solves eq. (5.26).

5.3 Pencil of cubics

Another way to describe a rational elliptic surface is as a pencil of cubics. For completeness, we describe our dP_9 in this framework.

A pencil of cubics is given by two homogeneous cubic polynomials $F(x, y, z)$ and $G(x, y, z)$. Their ratio $[F : G] \in \mathbb{P}^1$ is well defined apart from the 9 points[13] where $F = G = 0$. Hence, their ratio is not quite a function, but rather a rational[14] map $[F : G] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. However, after blowing up these 9 points in $\mathbb{P}^2$, we do get a map that is defined everywhere,

$$\mathrm{Bl}(\mathbb{P}^2) \longrightarrow \mathbb{P}^1 . \tag{5.29}$$

This blowup is our rational elliptic surface. Put differently, consider the hypersurface

$$B = \Big\{ \Big([x : y : z],\ [\mu : \nu] \Big) \Big| \ \mu F(x, y, z) + \nu G(x, y, z) = 0 \Big\} \subset \mathbb{P}^2 \times \mathbb{P}^1 . \tag{5.30}$$

Projecting to the $\mathbb{P}^2$ factor of the ambient space, we see that B is a blow-up of $\mathbb{P}^2$ at 9 points. Projecting to the $\mathbb{P}^1$, we see that B is elliptically fibered.

For example, take

$$\begin{aligned} F(x, y, z) &= \sqrt[3]{2}\Big(x^2 y + \omega y^2 z + \omega^2 z^2 x \Big) , \qquad \omega = e^{\frac{2\pi i}{3}} \\ G(x, y, z) &= -\frac{3 + i\sqrt{3}}{36} \Big(x^3 + y^3 + z^3 + 6xyz \Big) . \end{aligned} \tag{5.31}$$

The Weierstrass form of the cubic $F(x, y, z) + tG(x, y, z)$ is precisely eq. (5.27), corresponding to $\gamma = -\frac{1}{48}$ in our one parameter family of rational elliptic surfaces. This pencil of cubics has 9 distinct base points, which is to say that there are 9 solutions to $F = G = 0$, all with multiplicity 1.

One of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ generators is straightforward to write down:

$$g_1 : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^1, \Big([x : y : z],\ [\mu, \nu] \Big) \mapsto \Big([y : z : x],\ [\mu, \omega\nu] \Big) . \tag{5.32}$$

It preserves the hypersurface, $g_1(B) = B$, and, therefore, restricts to a $\mathbb{Z}_3$ action on B. Of course, we know that there must be another $\mathbb{Z}_3$ automorphism of B which commutes with eq. (5.32). However, within the description by a pencil of cubics, it is difficult to discuss the sections of the elliptic fibration. Recall that the $\mathbb{Z}_3$ generator is a translation by an order 3 section. Therefore, we must be able to explicitly write down the sections and the group action on them. This is much easier to do in the Weierstrass model.

[13]For simplicity, we assume here that the 9 points are distinct. This is the case in the pencil we are interested in.

[14]Rational maps are customarily denoted by a broken arrow $A \dashrightarrow B$.

6. Homology of the surface

6.1 Geometry of sections and singular fibers

Now that we have established the existence of a rational elliptic surface B with $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \subset \mathrm{Aut}(B)$, we want to compute the action of G on the homology group $H_2(B, \mathbb{Z})$. Using this, we can compute the Hodge numbers of our Calabi-Yau manifold. However, first we have to find one final piece of information. That is, we need to know how the sections σ, ξ, η, and the singular fibers intersect.

To do this, consider a section $\rho : \mathbb{P}^1 \to B$, $t \mapsto [\rho_x(t) : \rho_y(t) : 1]$. Again, globally, $\rho_x \in \Gamma\big(\mathfrak{M}(2)\big)$ and $\rho_y \in \Gamma\big(\mathfrak{M}(3)\big)$, that is, they are meromorphic functions of homogeneous degree 2 and 3 respectively. In general, ρ_x and ρ_y will have poles. There is nothing wrong with this since they correctly define homogeneous coordinates in $\mathbb{P}^2$. A pole only signifies that one should do a homogeneous rescaling, in which case the homogeneous coordinates will be finite. For certain sections, ρ_x and ρ_y will have no poles. In this case, ρ_x and ρ_y are actual polynomials of degree 2 and 3 respectively and homogeneous rescalings are not necessary. This implies that

$$[\rho_x(t) : \rho_y(t) : 1] \neq [0 : 1 : 0] \qquad \forall t \in \mathbb{P}^1 \ . \tag{6.1}$$

In other words, such sections do not intersect the 0-section σ. In fact, the converse is true. These are precisely the sections not intersecting σ. There are at most 240 such sections, their height pairing is $\langle \rho, \rho \rangle \leq 2$ and they generate the Mordell-Weil group. This follows again from the classification of A-D-E lattices and is proven in [56].

From the eqs. (5.28) and (5.24), it is obvious that the sections η, ξ do not intersect σ. It remains to understand where they intersect the singular fibers. Once this is known, we can compute their height pairings, eq. (4.15). In our case,[15] the only singular fibers with multiple irreducible components are the three I_3 fibers. Let $\Theta_{i,0}$ be the reducible component of the i^{th} I_3 singular fiber which intersects σ, where $i = 1, 2, 3$. Denote by $\Theta_{i,1}$ and $\Theta_{i,2}$ the other reducible components. The structure of the i^{th} singular I_3 fiber is shown in figure 10. Using eq. (5.28) for the section ξ, we can easily check that it passes through the singularity for 2 of the 3 I_3 singular fibers. (The surface B is smooth, but the Weierstrass model W_B contains singularities). Likewise, we can use eq. (5.24) to show that η passes through all 3 singular points, and so, never intersects $\Theta_{i,0}$, $i = 1, 2, 3$. Hence, the curves must intersect as in figure 11, up to relabeling of $\Theta_{i,j}$.

We can summarize the structure of the Mordell-Weil lattice as follows. The free part of the Mordell-Weil lattice is A_2^*, the dual of the A_2 root lattice (see figure 12). This lattice is generated by 2 points of minimal length $(\frac{2}{3})^{\frac{1}{2}}$. Actually, the 6 minimal lattice points are nothing else but the section ξ of eq. (5.28) and its images under α_B and $(-1)_B$ (the involution on B acting as -1 in each fiber separately). All height pairings can be simply computed from the lattice, figure 12.

To check this, first note that these 6 sections are indeed distinct and of the same height norm since the height pairing is preserved by α_B and $(-1)_B$. We have only to

[15] That is, the rational elliptic surface defined by the Weierstrass equation eq. (5.21). Here, of course, there are fewer than the aforementioned 240 sections not intersecting σ.

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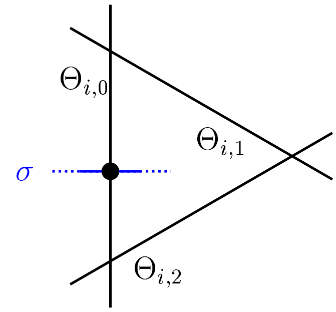

Figure 10: The i^{th} singular fiber of I_3 type.

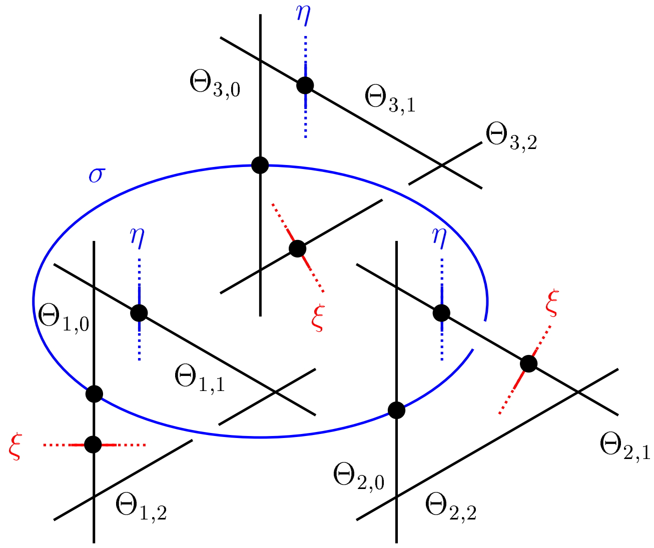

Figure 11: Sections and reducible fibers on B.

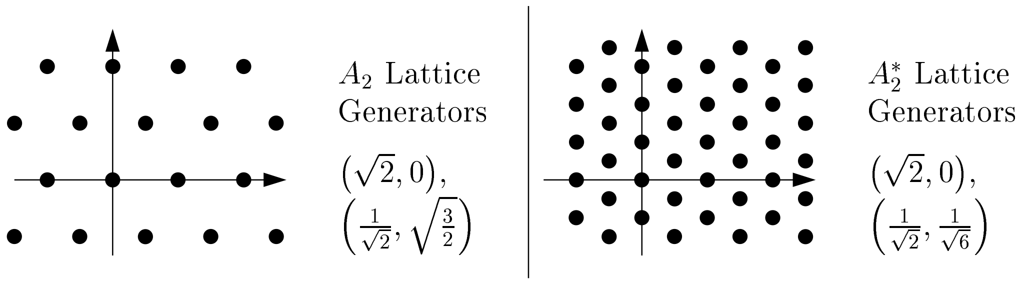

Figure 12: The A_2 and A_2^* lattice.

calculate $\langle \xi, \xi \rangle$. This can be done using the general formula in eq. (4.15). From this, we can immediately compute $\langle \xi, \xi \rangle$ to be

$$\langle \xi, \xi \rangle = 1 + 0 + 0 - (-1) - \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1}_{(1,1)} - \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1}_{(2,2)} = \frac{2}{3}\,. \tag{6.2}$$

Hence, ξ really is a minimal length lattice point. Finally, note that although we used the explicit equation for the sections to keep things simple, the intersection pattern in figure 11 is required by the height paring of ξ, η, and $\xi \boxplus \eta$ whenever ξ is a minimal length point of the Mordell-Weil lattice. This is explained in more detail in appendix C.

6.2 Action on homology

To determine the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ action on the homology group $H_2(B, \mathbb{Z})$, we first have to pick convenient generators. To do this, recall the description of the Mordell-Weil group via the exact sequence

$$0 \longrightarrow T \longrightarrow H_2(B, \mathbb{Z}) \longrightarrow E(K) \longrightarrow 0\,. \tag{6.3}$$

We can turn this description around and think of $H_2(B, \mathbb{Z})$ as an extension of T by $E(K)$. This implies that we can generate all of $H_2(B, \mathbb{Z})$ using images of generators of T and the lifts of generators of $E(K)$. The generators of T are naturally generators of the homology group, and the obvious lift of the sections is to take their homology classes.[16]

Hence, we choose the following set of generators for the homology group of our rational elliptic surface.

σ: The 0-section

F: A generic fiber.

$\Theta_{1,1}, \ldots, \Theta_{3,2}$: The 6 irreducible components of the three I_3 singular fibers not intersecting σ. We choose the indexing such that α_B cyclically permutes $\Theta_{1,i} \to \Theta_{2,i} \to \Theta_{3,i}$.

$\xi, \alpha_B \xi$: Two sections generating the free part of the Mordell-Weil group.

These form a basis for $H_2(B, \mathbb{Q})$, but they only generate an index 3 sublattice of $H_2(B, \mathbb{Z})$ as is clear from the short exact sequence above. To generate all of $H_2(B, \mathbb{Z})$, we have to include the homology class of the generator of $E(K)_{\mathrm{Tor}}$.

η: The α_B-invariant order 3 section η.

Of course, we have now chosen 11 generators. Since $\mathrm{rank} H_2(B, \mathbb{Z}) = 10$, there must be one relation between these. It is given by

$$\eta = \sigma + F - \frac{2}{3}\Big(\Theta_{1,1} + \Theta_{2,1} + \Theta_{3,1}\Big) - \frac{1}{3}\Big(\Theta_{1,2} + \Theta_{2,2} + \Theta_{3,2}\Big)\,. \tag{6.4}$$

This relation can easily be checked by computing intersection numbers[17] using figure 11. The intersection number of both sides with the 10 generators of $H_2(B, \mathbb{Q})$ is the same. Since there is no torsion in the integral homology group $H_2(B, \mathbb{Z})$, this implies the equality eq. (6.4). We could now eliminate one generator, for example $\Theta_{1,2}$, and have a basis for $H_2(B, \mathbb{Z})$. But then we would break the symmetry between the three I_3 fibers. Hence, we prefer to work with the 11 generators, even though one of them is redundant. For reference, we list their intersection matrix in table 6.

[16]Warning: for the rest of this paper, we make no distinction between a section and its homology class.

[17]The only possible complication is finding the intersection of the left side with ξ. This follows from the height pairing $\langle \eta, \xi \rangle = 0$ and eq. (4.15).

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$\cdot$	σ	F	$\Theta_{1,1}$	$\Theta_{2,1}$	$\Theta_{3,1}$	$\Theta_{1,2}$	$\Theta_{2,2}$	$\Theta_{3,2}$	ξ	$\alpha_B\xi$	η
σ	-1	1	0	0	0	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0	1	1	1
$\Theta_{1,1}$	0	0	-2	0	0	1	0	0	0	0	1
$\Theta_{2,1}$	0	0	0	-2	0	0	1	0	1	0	1
$\Theta_{3,1}$	0	0	0	0	-2	0	0	1	0	1	1
$\Theta_{1,2}$	0	0	1	0	0	-2	0	0	0	1	0
$\Theta_{2,2}$	0	0	0	1	0	0	-2	0	0	0	0
$\Theta_{3,2}$	0	0	0	0	1	0	0	-2	1	0	0
ξ	0	1	0	1	0	0	0	1	-1	1	0
$\alpha_B\xi$	0	1	0	0	1	1	0	0	1	-1	0
η	0	1	1	1	1	0	0	0	0	0	-1

Table 6: Intersection matrix of the homology generators.

Now that we have defined a set of generators for the homology group, we want to know how G acts on them. That is, we must determine the push forwards $(\alpha_B)_*$, $(t_\xi)_*$, and $(t_\eta)_*$. Most follow directly from the definition. The only tricky part is to find the action of $(t_\xi)_*$ and $(t_\eta)_*$ on the sections ξ, $\alpha_B\xi$, and η. In other words, we have to determine the homology classes of $\xi \boxplus \xi$, $\xi \boxplus \alpha_B\xi$, $\eta \boxplus \xi$, $\eta \boxplus \alpha_B\xi$, and $\eta \boxplus \eta$. We described the Mordell-Weil group $E(K)$ as a quotient of $H_2(B,\mathbb{Z})$ by T, see eq. (4.13). Therefore, the homology class of the Mordell-Weil sum $\mu \boxplus \nu$ is $\mu + \nu +$ (something in T). That is

$$\mu \boxplus \nu = \mu + \nu - \sigma + \left(\text{linear combination of } F, \Theta_{1,1}, \ldots, \Theta_{3,2}\right), \tag{6.5}$$

where we fixed the coefficient of σ by the intersection number with F using the fact that $s \cdot F = 1$ for any section s. We can compute the intersection number of any section with $\mu \boxplus \nu$ from the height pairing and the structure of the Mordell-Weil group. The intersection with the remaining homology generators $\Theta_{1,1}, \ldots, \Theta_{3,2}$ can simply be read off from figure 11. Determining the coefficients in eq. (6.5) is then a linear algebra problem. We find

$$\begin{aligned}
\xi \boxplus \xi &= \xi + \xi - \sigma + \Theta_{2,1} + \Theta_{3,2} && (6.6a)\\
\alpha_B\xi \boxplus \xi &= \alpha_B\xi + \xi - \sigma - F + \Theta_{3,1} + \Theta_{3,2} && (6.6b)\\
\eta \boxplus \xi &= \eta + \xi - \sigma - F + \Theta_{2,1} + \Theta_{3,1} + \Theta_{3,2} && (6.6c)\\
\eta \boxplus \alpha_B\xi &= \eta + \alpha_B\xi - \sigma - F + \Theta_{1,1} + \Theta_{1,2} + \Theta_{3,1} && (6.6d)\\
\eta \boxplus \eta &= \sigma + F - \frac{1}{3}\sum \Theta_{i,1} - \frac{2}{3}\sum \Theta_{i,2} \\
&= \eta + \eta - \sigma - F + \Theta_{1,1} + \Theta_{2,1} + \Theta_{3,1}\,. && (6.6e)
\end{aligned}$$

From this, we can determine the entire G action on $H_2(B,\mathbb{Z})$. The result is summarized in table 7. Instead of repeating the same arguments over and over again, we will only discuss a few representative cases:

$(\alpha_B)_*\sigma$: By definition α_B fixes the 0-section σ. Hence $(\alpha_B)_*\sigma = \sigma$.

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x	$(\alpha_B)_* x$	$(t_\xi)_* x$	$(g_1)_* x = \big(t_\xi \circ \alpha_B\big)_* x$	$(t_\eta)_* x = (g_2)_* x$
σ	σ	ξ	ξ	η
F	F	F	F	F
$\Theta_{1,1}$	$\Theta_{2,1}$	$\Theta_{1,1}$	$\Theta_{2,2}$	$\Theta_{1,2}$
$\Theta_{2,1}$	$\Theta_{3,1}$	$\Theta_{2,2}$	$F - \Theta_{3,1} - \Theta_{3,2}$	$\Theta_{2,2}$
$\Theta_{3,1}$	$\Theta_{1,1}$	$F - \Theta_{3,1} - \Theta_{3,2}$	$\Theta_{1,1}$	$\Theta_{3,2}$
$\Theta_{1,2}$	$\Theta_{2,2}$	$\Theta_{1,2}$	$F - \Theta_{2,1} - \Theta_{2,2}$	$F - \Theta_{1,1} - \Theta_{1,2}$
$\Theta_{2,2}$	$\Theta_{3,2}$	$F - \Theta_{2,1} - \Theta_{2,2}$	$\Theta_{3,1}$	$F - \Theta_{2,1} - \Theta_{2,2}$
$\Theta_{3,2}$	$\Theta_{1,2}$	$\Theta_{3,1}$	$\Theta_{1,2}$	$F - \Theta_{3,1} - \Theta_{3,2}$
ξ	$\alpha_B \xi$	$2\xi - \sigma +$ $+ \Theta_{2,1} + \Theta_{3,2}$	$\alpha_B\xi + \xi - \sigma -$ $- F + \Theta_{31} + \Theta_{32}$	$\eta + \xi - \sigma - F +$ $+ \Theta_{2,1} + \Theta_{3,1} + \Theta_{3,2}$
$\alpha_B \xi$	$-\alpha_B\xi - \xi +$ $+ 3\eta + \Theta_{1,1} +$ $+ \Theta_{2,1} + \Theta_{3,1}$	$\alpha_B\xi + \xi - \sigma - F +$ $+ \Theta_{3,1} + \Theta_{3,2}$	$-\alpha_B\xi + 2\sigma + 2F -$ $- \Theta_{1,1} - \Theta_{1,2} -$ $- \Theta_{3,1} - \Theta_{3,2}$	$\eta + \alpha_B\xi - \sigma - F +$ $+ \Theta_{1,1} + \Theta_{1,2} + \Theta_{3,1}$
η	η	$\eta + \xi - \sigma - F +$ $+ \Theta_{2,1} + \Theta_{3,1} + \Theta_{3,2}$	$\eta + \xi - \sigma -$ $- F + \Theta_{2,1} +$ $+ \Theta_{3,1} + \Theta_{3,2}$	$2\eta - \sigma - F +$ $+ \Theta_{1,1} + \Theta_{2,1} + \Theta_{3,1}$

Table 7: Summary of the G action on $H_2(B, \mathbb{Z})$.

$(t_\xi)_*\sigma$: Translating any section by ξ is just the sum in the Mordell-Weil group. This is then

$$(t_\xi)_*\sigma = \sigma \boxplus \xi = \xi\,. \tag{6.7}$$

$(\alpha_B)_*\Theta_{1,1}$: Since α_B cyclically permutes the I_3 singular fibers, the $\Theta_{1,1}$ component is just mapped to the $\Theta_{2,1}$ component: $(\alpha_B)_*\Theta_{1,1} = \Theta_{2,1}$.

$(t_\xi)_*\Theta_{3,1}$: The translation by ξ rotates the 3rd singular I_3 fiber according to figure 11. Therefore, it cyclically permutes the irreducible components $\Theta_{3,1} \to \Theta_{3,0} \to \Theta_{3,2} \to \Theta_{3,1}$. But $\Theta_{3,0}$ is not part of our chosen set of generators. We must eliminate it using the relation $F = \sum \Theta_{3,j}$. Hence, $(t_\xi)_*\Theta_{3,1} = \Theta_{3,0} = F - \Theta_{3,1} - \Theta_{3,2}$.

and so on.

6.3 Invariant homology

Now that we have identified the $G_1 = \{1, g_1, g_1^2\}$ and $G_2 = \{1, g_2, g_2^2\}$ actions on the homology lattice, it is a simple exercise in linear algebra to find the invariant sublattice. Picking a basis for $H_2(B, \mathbb{Z})$, we can express $(g_1)_*$ and $(g_2)_*$ as commuting 10×10 matrices. The invariant homology is then precisely the $+1$ eigenspace of these matrices, which is

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straightforward to compute. Both of them are 4-dimensional, and a convenient choice of generators is

$$H_2(B,\mathbb{Z})^{G_1} = \operatorname{span}_{\mathbb{Z}} \Big\{ F,\ -\sigma+\Theta_{2,1}+\eta,\ \Theta_{1,1}+\Theta_{3,1}+\Theta_{2,2},\ \Theta_{3,1}+\Theta_{3,2}+2\xi+\alpha_B\xi \Big\} \tag{6.8}$$

and

$$\begin{aligned} H_2(B,\mathbb{Z})^{G_2} = \operatorname{span}_{\mathbb{Z}} \Big\{ & F,\ \Theta_{1,1}+\Theta_{2,1}+\Theta_{3,1}+3\eta, \\ & -\Theta_{1,1}+\Theta_{3,1}-\Theta_{2,2}+\Theta_{3,2}+3\xi+3\alpha_B\xi, \\ & -\sigma+\Theta_{1,1}+\Theta_{2,1}+\Theta_{2,2}-\xi-2\alpha_B\xi+\eta \Big\}. \end{aligned} \tag{6.9}$$

We are, of course, interested in the $G = G_1 \times G_2$ invariant subspace. This is the intersection

$$H_2(B,\mathbb{Z})^G = H_2(B,\mathbb{Z})^{G_1} \cap H_2(B,\mathbb{Z})^{G_2}. \tag{6.10}$$

To prove this, we have to show that the inclusions "$\subseteq$" and "$\supseteq$" hold simultaneously. The first inclusion is true, since a G invariant homology class is necessarily invariant under the subgroups G_1 and G_2. For the inclusion in the other direction, note that every element $g \in G$ can be written as a product

$$g = g_1^{n_1} g_2^{n_2}, \qquad n_1, n_2 = 0,1,2. \tag{6.11}$$

Therefore, a homology class preserved by $(g_1)_*$ and $(g_2)_*$ is also preserved by $g_* = (g_1^{n_1} g_2^{n_2})_*$. Hence, to compute the G invariant homology we only have to intersect the G_1 invariant subspace with the G_2 invariant subspace. This is again simple linear algebra, and we only state the result. The $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ invariant homology group has rank 2 and is generated by

$$t_1 = F, \qquad t_2 = -\sigma+\Theta_{2,1}+\Theta_{3,1}+\Theta_{3,2}+2\xi+\alpha_B\xi+\eta-F. \tag{6.12}$$

That is,

$$H_2(B,\mathbb{Z})^G = t_1\mathbb{Z} \oplus t_2\mathbb{Z}. \tag{6.13}$$

7. The Calabi-Yau threefold

7.1 The fiber product

The fiber product of two dP_9 surfaces is a Calabi-Yau threefold, as already mentioned in section 3. We denote these two dP_9 surfaces by B_1 and B_2 and the Calabi-Yau threefold by $\widetilde{X}$. Moreover, we want a $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ action $\widetilde{X}$ and, hence, on B_1 and B_2 satisfying the constraints outlined in previous sections. Therefore, we choose B_1 and B_2 to each correspond to case 39a or one of the three special limits 39c, 68, and 68 in table 3. By definition, these surfaces come with projections

$$\beta_j : B_j \to \mathbb{P}^1, \qquad j = 1,2. \tag{7.1}$$

The fiber product $\widetilde{X}$ is defined as the hypersurface

$$\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2 = \Big\{ (p_1,p_2) \in B_1 \times B_2 \Big| \ \beta_1(p_1) = \beta_2(p_2) \Big\} \tag{7.2}$$

within $B_1 \times B_2$. The one equation in a $\dim_{\mathbb{C}}(B_1 \times B_2) = 4$ dimensional space defines a $\dim_{\mathbb{C}}\left(\widetilde{X}\right) = 3$ dimensional hypersurface, as desired. Furthermore, as was shown in [60], the first Chern class is

$$c_1\left(\widetilde{X}\right) = 0\,. \tag{7.3}$$

Hence, $\widetilde{X}$ is a Calabi-Yau threefold.

There is the following subtlety in this construction. If we consider only one of the dP_9 surfaces, say B_1, then the choice of coordinates on the base $\mathbb{P}^1$ does not matter. That is, pick $PGL(2,\mathbb{C}) \ni \tau : \mathbb{P}^1 \to \mathbb{P}^1$. Then $\beta_1 : B_1 \to \mathbb{P}^1$ and $\tau \circ \beta_1 : B_1 \to \mathbb{P}^1$ are two different projections. But, since they are isomorphic, it makes little sense to distinguish them. However, we are considering two dP_9 surfaces simultaneously. Changing the coordinates on the two base $\mathbb{P}^1$s relative to one another does make a difference for the fiber product. This is clear from the following description of the fiber product. The fiber product $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$ is a T^4 fibration over $\mathbb{P}^1$, where the fiber over $s \in \mathbb{P}^1$ is precisely $\beta_1^{-1}(s) \times \beta_2^{-1}(s)$. Changing β_1 to $\tau \circ \beta_1$ then changes which fibers of B_1 and B_2 are paired up, so it changes the fiber product.

Therefore, we must be careful with the relative choice of coordinates implicit in the projections β_1 and β_2. To accomplish our goal, we must choose the projections so that

- the G action extends to the fiber product, that is, the hypersurface $\widetilde{X} \subset B_1 \times B_2$ is preserved and
- the G action is free on $\widetilde{X}$, that is, the hypersurface is disjoint from the fixed point set in $B_1 \times B_2$.

Let us first discuss under what conditions the G action extends to $\widetilde{X}$. Recall that we have two generators, g_1 and g_2, where g_1 rotates the base $\mathbb{P}^1$ while g_2 does not. Since g_2 keeps every fiber stable, its action always extends to the fiber product. On the other hand side, g_1 moves the fibers of B_1 and B_2. Therefore, we must ensure that the fibers paired up in the fiber product stay together under the g_1 action. That is, if

$$F_j \subset B_j\,, \qquad j = 1, 2 \tag{7.4}$$

are two fibers, then

$$\beta_1(F_1) = \beta_2(F_2) \quad \Rightarrow \quad \beta_1\Big(g_1(F_1)\Big) = \beta_2\Big(g_1(F_2)\Big)\,. \tag{7.5}$$

This means that the induced action on the base $\mathbb{P}^1$ must be the same. In particular, the two fixed points on the base $\mathbb{P}^1$ must be the same. We take these two fixed points to be

$$[0:1],\ [1:0] \in \mathbb{P}^1\ . \tag{7.6}$$

Henceforth, we require that the induced action on the base $\mathbb{P}^1$ is the same, that is

$$\beta_1 \circ g_1 \circ \beta_1^{-1} = \beta_2 \circ g_1 \circ \beta_2^{-1}\,. \tag{7.7}$$

As we have seen, this implies that β_j projects the two g_1 stable fibers of B_j down to the two special points in eq. (7.6) for $j = 1, 2$.

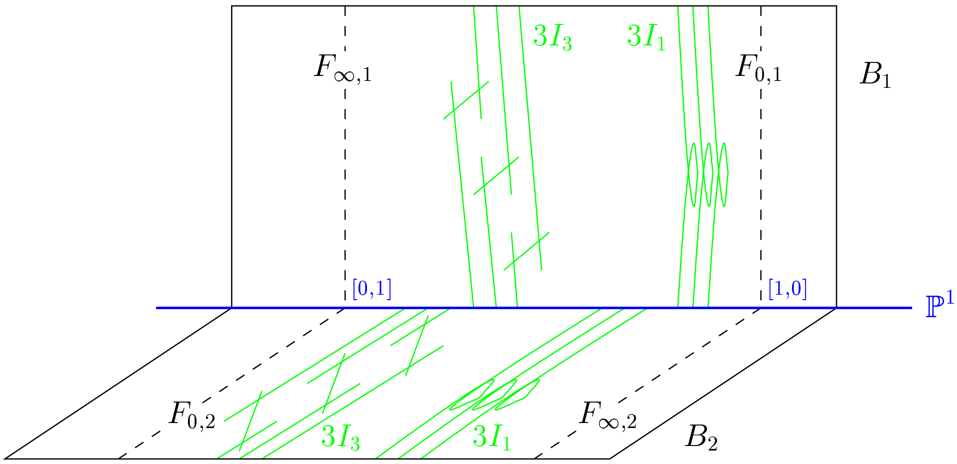

Figure 13: The fiber product $B_1 \times_{\mathbb{P}^1} B_2$, showing our identifictaion of fibers.

Therefore, we have chosen the projections β_1 and β_2 so that the G action extends to $\widetilde{X}$. However, this does not fix the projections uniquely. We still have both a continuous and a discrete choice, which we will now use to obtain a free G action. Recall that G acts freely on $\widetilde{X}$ if and only if for every subgroup of G there are never two fibers containing fixed points which are paired in the fiber product. That is, for each fiber $\beta_1^{-1}(s) \times \beta_2^{-1}(s)$ of $\widetilde{X}$ and for each $g \in G$, at most one of the fibers $\beta_1^{-1}(s)$ of B_1 and $\beta_2^{-1}(s)$ of B_2 contains a point fixed by g. We have to distinguish the cases where s is or is not one of the special points eq. (7.6). First, assume that $s \neq [0:1], [1:0]$. Furthermore, assume that $\beta_2^{-1}(s)$ contains a fixed point of $g \in G$. Then, we must show that we can choose β_1 such that $\beta_1^{-1}(s)$ does not have a g fixed point. This can be achieved using the remaining continuous rescaling

$$\mathbb{P}^1 \to \mathbb{P}^1, \; [z_0 : z_1] \mapsto [\lambda z_0 : z_1], \qquad \lambda \in \mathbb{C} - \{0\}. \tag{7.8}$$

Since there are only a finite number of special fibers $\beta_2^{-1}(s)$, one can always find a suitable rescaling λ. Hence, without loss of generality, we assume that β_1 and β_2 are chosen so that the fiber product $\widetilde{X}$ does not have any fixed points except, possibly, over $[0:1], [1:0]$. To exclude fixed points in these two fibers, note that our dP_9 surfaces B_j, $j = 1, 2$ were constructed so that for every $g \in G$ there is never a g fixed point in both $\beta_j^{-1}([0:1])$ and $\beta_j^{-1}([1:0])$ simultaneously. Hence, we can use the remaining discrete choice in the $\mathbb{P}^1$ coordinates to ensure that no fibers with fixed points are paired up. This remaining choice is to exchange the homogeneous coordinates on $\mathbb{P}^1$, thus exchanging $[0:1]$ and $[1:0]$. Doing this, if necessary, ensures that there are no fixed points in $\widetilde{X}$ over $[0:1], [1:0]$ as well. The structure of such a fiber product is illustrated in figure 13.

To summarize, one can always choose the projections $\beta_j : B_j \to \mathbb{P}^1$ in such a way that G acts freely on the fiber product $\widetilde{X}$. In the following, we will assume that this is the case. By construction, this Calabi-Yau threefold $\widetilde{X}$ is elliptically fibered with respect to each of the two projections

$$\pi_j : \widetilde{X} \to B_j \,, \qquad j = 1, 2 \,. \tag{7.9}$$

7.2 Homology of the fiber product

We have constructed a specific family of Calabi-Yau threefolds

$$\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2 \,. \tag{7.10}$$

Now, we want to determine its homology and Hodge numbers. We refer the reader to [60] to explicit proofs.

First, note that the fiber product is simply connected

$$\pi_1\big(\widetilde{X}\big) = 1 \,. \tag{7.11}$$

Furthermore, $\widetilde{X}$ can be glued from T^2 bundles as follows. Recall that $\widetilde{X}$ can be thought of as a $T^2 \times T^2$ fibration over $\mathbb{P}^1$. Choose a sufficiently fine covering of $\mathbb{P}^1$. Then, locally, one of the two possible T^2 fibrations is actually smooth. This implies the vanishing of the Euler characteristic

$$\chi\big(\widetilde{X}\big) = 0 \,. \tag{7.12}$$

Moreover, it can be shown that the second cohomology group $H^2\big(\widetilde{X}, \mathbb{Z}\big) \simeq \mathrm{Pic}\,\big(\widetilde{X}\big)$ is

$$H^2\Big(\widetilde{X}, \mathbb{Z}\Big) \simeq \frac{H^2(B_1,\mathbb{Z}) \oplus H^2(B_2,\mathbb{Z})}{H^2(\mathbb{P}^1,\mathbb{Z})} \,, \tag{7.13}$$

Counting dimensions, that is, ignoring torsion, and using eq. (4.4) we immediately find that

$$H^2\Big(\widetilde{X}, \mathbb{Q}\Big) = \mathbb{Q}^{19} \,. \tag{7.14}$$

This determines the Hodge diamond as follows. The second Betti number is

$$b_2 = h^{0,2} + h^{1,1} + h^{2,0} = 19 \,. \tag{7.15}$$

For a Calabi-Yau threefold, $h^{0,2}$ and $h^{2,0}$ always vanish. Therefore, $h^{1,1} = 19$. Finally, the Euler characteristic of any Calabi-Yau threefold is $2(h^{1,1} - h^{2,1})$. In our case, the Euler characteristic vanishes and, hence, $h^{1,1} = h^{2,1}$. To summarize, the Hodge diamond of $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$ is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 19 & & 0 & \\ 1 & & 19 & & 19 & & 1 \\ & 0 & & 19 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \,. \tag{7.16}$$

7.3 Homology of the Calabi-Yau manifold X

Having constructed the simply connected Calabi-Yau manifold $\widetilde{X}$ with a $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ action, we now define

$$X = \widetilde{X}/G \,. \tag{7.17}$$

Because the G action is free and analytic, this quotient is again a smooth Kähler manifold. In fact, there is a general theorem[18] that a fixed point free quotient of a Calabi-Yau threefold is again a Calabi-Yau threefold.

The Calabi-Yau threefold X clearly has fundamental group

$$\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3 \,. \tag{7.18}$$

Moreover, its Euler characteristic is easily computed to be

$$\chi\Big(\widetilde{X}/G\Big) = \frac{1}{|G|}\chi\Big(\widetilde{X}\Big) = \frac{1}{9}\cdot 0 = 0 \,. \tag{7.19}$$

The individual Betti and Hodge numbers can be found as follows. In general, the rational cohomology groups on X are the invariant cohomology groups on $\widetilde{X}$. That is,

$$H^*\big(X, \mathbb{Q}\big) = H^*\big(\widetilde{X}, \mathbb{Q}\big)^G \,. \tag{7.20}$$

It follows from eq. (7.13) that

$$H^2\big(\widetilde{X}, \mathbb{Q}\big)^G = \left(\frac{H^2(B_1, \mathbb{Q}) \oplus H^2(B_2, \mathbb{Q})}{H^2(\mathbb{P}^1, \mathbb{Q})}\right)^G = \frac{H^2(B_1, \mathbb{Q})^G \oplus H^2(B_2, \mathbb{Q})^G}{H^2(\mathbb{P}^1, \mathbb{Q})} \,. \tag{7.21}$$

The invariant cohomology of each dP_9 surfaces is Poincaré dual to the invariant homology given in eq. (6.13). By counting dimensions, we compute the second Betti number of X to be $b_2 = 2 + 2 - 1 = 3$. By the same argument as in the previous section, this determines all Hodge numbers of the Calabi-Yau threefold X. The Hodge diamond of $X = \widetilde{X}/G$, therefore, is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \,. \tag{7.22}$$

Note that there are $h^{2,1} = 3$ complex structure moduli. These can be understood as follows. Recall that each of the two surfaces B_1 and B_2 come with a single parameter. This accounts for 2 of the 3 moduli. The third modulus is the relative scaling that is implicit in the projections $\beta_j : B_j \to \mathbb{P}^1$, see eq. (7.8).

Acknowledgments

Volker Braun and Burt Ovrut are supported in part by the NSF Focused Research Grant DMS 0139799 the DOE under contract No. DE-AC02-76-ER-03071. Tony Pantev is supported in part by NSF grants DMS 0099715, DMS 0139799, and an A. P. Sloan Research Fellowship. René Reinbacher is supported by the Department of Physics and Astronomy of Rutgers University under grant No. DOE-DE-FG02-96ER40959.

[18] This is not obvious, and only holds for proper Calabi-Yau threefolds. Naively, one might fear that the first Chern class of the quotient is a torsion cohomology class. Equivalently, the holomorphic $(3,0)$ form Ω might not be invariant. A proof of the theorem can be found in [61].

Parameter	Singular fibers
generic	$3I_0,\, 3I_3$
$\gamma = -\frac{1}{24}$	$3I_0,\, I_9$
$\gamma = -\frac{5}{216}$	$4I_3$
$\gamma \to \infty,\, t\gamma^{-\frac{1}{3}}$ fixed	$4IV$

Table 8: Singular fibers of the 1-parameter family 39a and limits thereof.

A. Limits of the Weierstrass equation

The resolution of the zero set of a Weierstrass equation

$$y^2 z = x^3 + a(t)xz^2 + b(t)z^3 \tag{A.1}$$

is a rational elliptic surface. It is well-known how to read off the singular fibers. The fibration degenerates whenever the discriminant $D = 4a(t)^3 + 27b(t)^2$ vanishes. At each such zero, the type of Kodaira fiber is determined by the order of vanishing of the discriminant and the coefficients as in table 5.

For our one parameter Weierstrass equation (5.21), the singular fibers will change at special points of the parameter γ. Hence, the Mordell-Weil group also changes. We find the singular fibers presented in table 8.

B. Orbifold resolutions

Let G be an arbitrary finite group acting on a rational elliptic surface $B \to \mathbb{P}^1$. Then the quotient B/G has orbifold singularities. We want to show that the minimal resolution $\widehat{B/G}$ is again a rational elliptic surface. This relies on the classification of algebraic surfaces. We review here some fundamental facts.

The plurigenera $P_n(X)$ of any surface X are the number of sections of a certain line bundle,

$$P_n(X) = \dim H^0\Big(\mathcal{O}_X[nK]\Big)\,. \tag{B.1}$$

Their asymptotic growth with $n \in \mathbb{Z}$, $n \geq 0$ is the most important birational invariant of the surface X. It is called the Kodaira dimension and takes values in

$$\kappa(X) \in \{-\infty, 0, 1, 2\}\,. \tag{B.2}$$

By definition of κ, the plurigenera P_n grow like n^κ. What is the Kodaira dimension of a rational elliptic surface B? Recall that B is the blow up of $\mathbb{P}^2$ at 9 points. Since the Kodaira dimension is a birational invariant, we may just as well compute the Kodaira dimension of $\mathbb{P}^2$. But all plurigenera

$$P_n\big(\mathbb{P}^2\big) = 0\,. \tag{B.3}$$

Hence, the Kodaira dimensions are

$$\kappa(B) = \kappa\big(\mathbb{P}^2\big) = -\infty\,. \tag{B.4}$$

Now, the G action preserves the line bundle $\mathcal{O}_B[nK]$ and, therefore, defines an action on $H^0\big(\mathcal{O}_B[nK]\big)$. The plurigenera of the quotient are the dimensions of the invariant parts

$$P_n\big(\widehat{B/G}\big) = \dim H^0\Big(\mathcal{O}_B[nK]\Big)^G \leq \dim H^0\Big(\mathcal{O}_B[nK]\Big) = 0\,. \tag{B.5}$$

Therefore,

$$\kappa\big(\widehat{B/G}\big) = -\infty\,. \tag{B.6}$$

Another birational invariant is the irregularity q. In our case,

$$q\big(\widehat{B/G}\big) = \dim H^0\big(\Omega^1\big)^G = 0 = \dim H^0\big(\Omega^1\big) = q(\mathbb{P}^2) = q(B)\,. \tag{B.7}$$

By the classification of algebraic surfaces, a surface of Kodaira dimension $\kappa = -\infty$ and irregularity $q = 0$ is rational. Furthermore, the G action has to preserve the elliptic fibration of B, as proven in section 3. Hence, the quotient B/G and its resolution $\widehat{B/G}$ are also elliptically fibered. Therefore, $\widehat{B/G}$ is again a rational elliptic surface.

C. A synthetic approach

While we argued that one can read off the intersection pattern in figure 11 from the Weierstrass equation, it can actually be determined on general grounds without having to refer to any geometric realization. We need only know that

- The rational elliptic surface B has $3I_3$ and $3I_1$ singular fibers.
- There exists an order 3 section $\eta \in E(K)$, yielding a $\mathbb{Z}_3$ action $t_\eta : B \to B$.
- There exists another $\mathbb{Z}_3$ action α_B on B, fixing the 0-section and acting non-trivially on the base $\mathbb{P}^1$.
- α_B has isolated fixed points on $\beta^{-1}([0,1])$ and, in addition, fixes $\beta^{-1}([1,0])$ pointwise.
- t_η and α_B commute.

We know the action of α_B on all homology generators except on the free part of the Mordell-Weil group. Now, the Euler characteristic of the α_B fixed point set is $\chi(B^{\alpha_B}) = 3$. The Lefschetz fixed point formula then determines the entire α_B action. The result is that $(\alpha_B)_*$ rotates $E(K)_{\text{free}} = A_2^*$ by 120^0. We can now pick any section ξ and define $g_1 = t_\xi \circ \alpha_B$. Together with $g_2 = t_\eta$ this generates a $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action on B. All g_1 fixed points are contained in $\beta^{-1}([0,1])$ if only $\xi_{[1,0]} \neq 0$. We ensure this by choosing a section ξ which does not intersect the 0-section, $\xi\sigma = 0$. The section ξ is then a minimal length point of the A_2^* lattice. Now, g_2 necessarily fixes the singular point on the I_1 fibers. Therefore, the quotient has a singularity modeled on $\mathbb{C}^2/\mathbb{Z}_3$. Resolving this contributes $3 \cdot 2 = 6$ to the Euler characteristic, so the only way to get $\chi(\widehat{B/G_2}) = 12$ is if g_2 acts freely on every other fiber. To summarize: g_1 fixes 3 points in $\beta^{-1}([0,1])$ and g_2 fixes one point in each of the three I_1 singularities.

Now, assume that η intersects the neutral component of the i^{th} I_3 fiber. Then, there must be at least 3 fixed points in this fiber, but there are actually none. Therefore, η cannot intersect $\Theta_{i,0}$. We choose our notation so that η intersects $\Theta_{i,1}$.

Finally, using the height pairing, we obtain

$$\frac{2}{3} = \langle \xi, \xi \rangle = 2 - \sum_s \operatorname{contr}_s(\xi, \xi)\,. \tag{C.1}$$

Since $\operatorname{contr}_s(\xi, \xi)$ is either 0 or $\frac{2}{3}$, we see that ξ has to intersect precisely one of the I_3 fibers in the neutral component, which we call $\Theta_{1,0}$. By cyclic symmetry (that is, the α_B action), ξ must intersect the other two I_3 fibers such that

- in one I_3 fiber, ξ and η intersect the same irreducible component and
- in the last I_3 fiber, ξ and η intersect different irreducible components.

We fix our notation completely by asking that ξ intersects $\Theta_{2,1}$ and $\Theta_{3,2}$. To summarize, we obtained precisely the intersection pattern in figure 11.

D. α_B-invariant homology

Although we do not need the α_B invariant part of the homology group in this paper, we record it here for future use. We already know the action of α_B on the homology, so it is again straightforward to identify the $+1$ eigenspace. The invariant sublattice has rank 4 and is generated by

$$H_2(B, \mathbb{Z})^{\alpha_B} = \operatorname{span}\Big(\sigma,\ F,\ \eta,\ \sum_{i=1}^{3} \Theta_{i,1}\Big)\,. \tag{D.1}$$

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